BIRKHÄUSER

Progress in Mathematics
264

Rosa M. Miró-Roig

Determinantal Ideals

INSTITUT D'ESTUDIS CATALANS MCMVII

Ferran Sunyer i Balaguer
Award winning monograph

BIRKHÄUSER

Progress in Mathematics
Volume 264

Rosa M. Miró-Roig

Determinantal Ideals

Birkhäuser
Basel · Boston · Berlin

Author:

Rosa M. Miró-Roig
Departament d'Algebra i Geometria
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 Barcelona
Spain
e-mail: miro@ub.edu

2000 Mathematics Subject Classification: primary 14M06; secondary 13C40, 13D45, 14F05

Library of Congress Control Number: 2007933915

Bibliographic information published by Die Deutsche Bibliothek. Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de

ISBN 978-3-7643-8534-7 Birkhäuser Verlag AG, Basel · Boston · Berlin

Basel · Boston · Berlin
P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany

ISBN 978-3-7643-8534-7 e-ISBN 978-3-7643-8535-4

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

Ferran Sunyer i Balaguer (1912–1967) was a self-taught Catalan mathematician who, in spite of a serious physical disability, was very active in research in classical mathematical analysis, an area in which he acquired international recognition. His heirs created the Fundació Ferran Sunyer i Balaguer inside the Institut d'Estudis Catalans to honor the memory of Ferran Sunyer i Balaguer and to promote mathematical research.
Each year, the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans award an international research prize for a mathematical monograph of expository nature. The prize-winning monographs are published in this series. Details about the prize and the Fundació Ferran Sunyer i Balaguer can be found at

`http://ffsb.iec.cat`

This book has been awarded the
Ferran Sunyer i Balaguer 2007 prize.

The members of the scientific commitee
of the 2007 prize were:

Antonio Córdoba
Universidad Autónoma de Madrid

Paul Malliavin
Université de Paris VI

Joseph Oesterlé
Université de Paris VI

Oriol Serra
Universitat Politècnica de Catalunya, Barcelona

Alan Weinstein
University of California at Berkeley

Ferran Sunyer i Balaguer Prize winners since 1998:

1998 Juan J. Morales-Ruiz
Differential Galois Theory and Non-integrability of Hamiltonian Systems, PM 179

1999 Patrick Dehornoy
Braids and Self-Distributivity, PM 192

2000 Juan-Pablo Ortega and Tudor Ratiu
Hamiltonian Singular Reduction, PM 222

2001 Martin Golubitsky and Ian Stewart
The Symmetry Perspective, PM 200

2002 André Unterberger
Automorphic Pseudodifferential Analysis and Higher Level Weyl Calculi, PM 209

Alexander Lubotzky and Dan Segal
Subgroup Growth, PM 212

2003 Fuensanta Andreu-Vaillo, Vincent Caselles and José M. Mazón
Parabolic Quasilinear Equations Minimizing Linear Growth Functionals, PM 223

2004 Guy David
Singular Sets of Minimizers for the Mumford-Shah Functional, PM 233

2005 Antonio Ambrosetti and Andrea Malchiodi
Perturbation Methods and Semilinear Elliptic Problems on R^n, PM 240

José Seade
On the Topology of Isolated Singularities in Analytic Spaces, PM 241

2006 Xiaonan Ma and George Marinescu
Holomorphic Morse Inequalities and Bergman Kernels, PM 254

2007 Rosa Miró-Roig
Determinantal Ideals, PM 264

To Pere and Rosa Maria
and
Joan, Maria and Joan, jr.

This page intentionally left blank

Contents

Introduction xi

1 Background

1.1 Minimal free resolutions, ACM schemes, and AG schemes 1

1.2 Determinantal ideals 13

1.3 CI-liaison and G-liaison 21

2 CI-liaison and G-liaison of Standard Determinantal Ideals

2.1 CI-liaison class of Cohen–Macaulay codimension 2 ideals 30

2.2 CI-liaison class of standard determinantal ideals 33

2.3 G-liaison class of standard determinantal ideals 41

3 Multiplicity Conjecture for Standard Determinantal Ideals

3.1 The multiplicity conjecture for Cohen–Macaulay codimension 2 ideals 47

3.2 The multiplicity conjecture for standard determinantal ideals . . . 50

4 Unobstructedness and Dimension of Families of Standard Determinantal Ideals

4.1 Families of codimension 2 Cohen–Macaulay algebras 65

4.2 Unobstructedness and dimension of families of determinantal schemes 67

5 Determinantal Ideals, Symmetric Determinantal Ideals, and Open Problems

5.1 Liaison class of determinantal and symmetric determinantal ideals 106

5.2 The multiplicity conjecture for determinantal and symmetric determinantal ideals 111

5.3 Unobstructedness and dimension of families of determinantal and symmetric determinantal ideals 119

Bibliography 129

Index 135

This page intentionally left blank

Introduction

In this work, we will deal with *standard determinantal ideals*, *determinantal ideals*, and *symmetric determinantal ideals*, i.e., ideals generated by the maximal minors of a homogeneous polynomial matrix, by the minors (not necessarily maximal) of a homogeneous polynomial matrix, and by the minors of a homogeneous symmetric polynomial matrix, respectively. Some classical ideals that can be constructed in this way are the homogeneous ideal of Segre varieties, the homogeneous ideal of rational normal scrolls, and the homogeneous ideal of Veronese varieties.

Standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals have been a central topic in both commutative algebra and algebraic geometry and they also have numerous connections with invariant theory, representation theory and combinatorics. Due to their important role, their study has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results are due to J.A. Eagon and M. Hochster [20] and to J.A. Eagon and D.G. Northcott [21]. J.A. Eagon and M. Hochster proved that generic determinantal ideals are Cohen–Macaulay while the Cohen–Macaulayness of symmetric determinantal ideals was proved by R. Kutz in [62, Theorem 1]. J.A. Eagon and D.G. Northcott constructed a finite free resolution for any standard determinantal ideal and as a corollary they got that standard determinantal ideals are Cohen–Macaulay. In [85], B. Sturmfels uses the Knuth–Robinson–Schensted (KRS) correspondence for the computation of Gröbner bases of determinantal ideals. The application of the KRS correspondence to determinantal ideals has also been investigated by S.S. Abhyankar and D.V. Kulkarni in [1] and [2]. Furthermore, variants of the KRS correspondence can be used to study symmetric determinantal ideals (see [17]) or ideals generated by Pfaffians of skew symmetric matrices (see [47], [5], and [18]). Many other authors have made important contributions to the study of standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals without even being mentioned here and we apologize to those whose work we may have failed to cite properly.

In this book, we will mainly restrict our attention to standard determinantal ideals and we will attempt to address the following three crucial problems.

(1) CI-liaison class and G-liaison class of standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals.

(2) The multiplicity conjecture for standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals.

(3) Unobstructedness and dimension of families of standard determinantal schemes, determinantal schemes, and symmetric determinantal schemes.

Given the extensiveness of the subject, it is not possible to go into great detail in every proof. Still, it is hoped that the material that we choose will be beneficial and illuminating for the reader. The reader can refer [10], [54], [56], [59], [60], [70], [9], and [22] for background, history, and a list of important papers.

Let us now briefly describe the contents of each single chapter of this book. We start out in Chapter 1 by fixing notation and providing the basic concepts in the field. Minimal free resolutions, arithmetically Cohen–Macaulay (ACM) schemes, and arithmetically Gorenstein (AG) schemes are the topics of Section 1.1. In Section 1.2, we recall the definition and basic facts on standard determinantal ideals, determinantal ideals and symmetric determinantal ideals, we also provide the results on good and standard determinantal schemes $X \subset \mathbb{P}^n$ and the associated complexes, used later on. In particular, we recall the definition of generalized Koszul complexes which provide minimal free R-resolutions of the homogeneous ideal $I(X)$ of X and of the canonical module K_X of X. Section 1.3 is devoted to the definition of CI-liaison and G-liaison and to overview the known results on liaison theory needed later on. In particular, we present G-liaison theory as a theory of divisors on arithmetically Cohen–Macaulay schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on a complete intersection scheme. In order for meaningful applications of G-liaison to be found, we need useful constructions of Gorenstein ideals. We end this chapter describing the method that has been used either directly or at least indirectly in most of the results about G-liaison discovered in the last years (see Theorems 1.3.11 and 1.3.12).

Chapter 2 is devoted to study the CI-liaison class and G-liaison class of standard determinantal ideals. Liaison theory has its roots dating more than a century ago although the greatest activity has been in the last 30 years, beginning with the work of C. Peskine and L. Szpiro [75], where they established liaison theory as a modern discipline and they gave a rigorous proof of Gaeta's theorem. The goal of Section 2.1 is to sketch a proof of Gaeta's theorem: every arithmetically Cohen–Macaulay codimension 2 subscheme X of $\mathbb{P}^n$ can be CI-linked in a finite number of steps to a complete intersection subscheme; i.e., X is licci. Since it is well known for subschemes of codimension 2 of $\mathbb{P}^n$ that arithmetically Cohen–Macaulay subschemes are standard determinantal (Hilbert–Burch theorem) and that arithmetically Gorenstein subschemes are complete intersections, Gaeta's theorem can be viewed as a first result about the CI-liaison and G-liaison of standard determinantal schemes. In Section 2.2, we prove that in the CI-liaison context Gaeta's theorem does not generalize well to subschemes $X \subset \mathbb{P}^n$ of higher codimension. More precisely, we introduce some graded modules which are CI-liaison invariants and we use them to prove the existence of infinitely many different CI-liaison classes containing standard determinantal curves $C \subset \mathbb{P}^4$ (see Remark 2.2.14). The purpose of Section 2.3 is to extend Gaeta's theorem, viewed as a statement on standard determinantal subschemes of codimension 2, to arbitrary codimension and to prove that any standard determinantal subscheme X of $\mathbb{P}^n$ can be G-linked in a finite number of steps to a complete intersection subscheme; i.e., X is glicci.

In Chapter 3, we study the relation between the graded Betti numbers of a homogeneous standard determinantal ideal $I \subset R = K[x_1, \dots, x_n]$ and the multiplicity of I, $e(R/I)$. The motivation for comparing the multiplicity of a graded R-module M to products of the shifts in a graded minimal free R-resolution of M comes from a paper of C. Huneke and M. Miller [51]. They focused on homogeneous Cohen–Macaulay ideals $I \subset R$ of codimension c with a pure resolution,

$$0 \longrightarrow \oplus R(-d_c)^{a_c} \longrightarrow \cdots \longrightarrow R(-d_1)^{a_1} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

and they proved for such ideals the following beautiful formula for the multiplicity of R/I:

$$e(R/I) = \frac{\prod_{i=1}^{c} d_i}{c!}.$$

Since then there has been a considerable effort to bound the multiplicity of a homogeneous Cohen–Macaulay ideal $I \subset R$ in terms of the shifts in its graded minimal free R-resolution; and J. Herzog, C. Huneke, and H. Srinivasan have made the following conjecture (*multiplicity conjecture*) which relates the multiplicity $e(R/I)$ to the maximum and the minimum degree shifts in the graded minimal free R-resolution of R/I.

Conjecture 0.0.1. *Let $I \subset R$ be a graded Cohen–Macaulay ideal of codimension c. We consider the minimal graded free R-resolution of R/I:*

$$0 \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Set

$$m_i(I) := \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$$

and

$$M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}.$$

Then, we have

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}.$$

There is a growing body of the literature proving special cases of Conjecture 0.0.1. For example, it holds for complete intersection ideals [46], powers of complete intersection ideals [37], perfect ideals with a pure resolution (i.e., $m_i = M_i$) [51], perfect ideals with a quasi-pure resolution (i.e., $m_i \geq M_{i-1}$) [46], perfect ideals of codimension 2 [46], and Gorenstein ideals of codimension 3 [67]. We devote

Chapter 3 to prove that Conjecture 0.0.1 works for standard determinantal ideals of arbitrary codimension (see Theorem 3.2.6). We end Chapter 3, proving that the ith total Betti number $\beta_i(R/I)$ of a standard determinantal ideal I can be bounded above by a function of the maximal shifts $M_i(I)$ in the minimal graded free R-resolution of R/I as well as bounded below by a function of both the maximal shifts $M_i(I)$ and the minimal shifts $m_i(I)$.

Hilbert schemes are not just sets of objects; they are endowed with a scheme structure far of being well understood. In Chapter 4, we consider some aspects related to families of standard determinantal schemes; more precisely, we address the problem of determining the unobstructedness and the dimension of families of standard (resp., good) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c. The first important contribution to this problem is due to G. Ellinsgrud [25]; in 1975, he proved that every arithmetically Cohen–Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is unobstructed (i.e., the corresponding point in the Hilbert scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+2})$ is smooth) provided $n \geq 1$, and he also computed the dimension of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+2})$ at $[X]$. Recall also that the homogeneous ideal of an arithmetically Cohen–Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is given by the maximal minors of a $t \times (t+1)$ homogeneous matrix, the Hilbert–Burch matrix; i.e., X is standard determinantal. The purpose of this chapter is to extend Ellingsrud's theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary codimension. We also address the problem whether the closure of the locus of standard determinantal schemes in $\mathbb{P}^{n+c}$ is an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$, and when $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is generically smooth along the determinantal locus (see Corollaries 4.2.37, 4.2.41, 4.2.43, and 4.2.44).

Given integers $b_1, \ldots, b_t$ and $a_0, a_1, \ldots, a_{t+c-2}$, we denote by

$$W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$$

the locus of good determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension $c \geq 2$ defined by the maximal minors of a homogeneous matrix $\mathcal{A} = (f_{ji})^{i=1,\ldots,t}_{j=0,\ldots,t+c-2}$, where $f_{ji} \in K[x_0, x_1, \ldots, x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. In Section 4.2, using induction on c by successively deleting columns of the largest possible degree and using repeatedly the Eagon–Northcott complexes and the Buchsbaum–Rim complexes associated with a standard determinantal scheme, we state an upper bound for the dimension of $W(\underline{b}; \underline{a})$ in terms of $b_1, \ldots, b_t$ and $a_0, a_1, \ldots, a_{t+c-2}$ (cf. Theorem 4.2.7 and Proposition 4.2.15). Using again induction on the codimension and the theory of Hilbert flag schemes, we analyze when the upper bound of $\dim W(\underline{b}; \underline{a})$, given in Theorem 4.2.7 and Proposition 4.2.15, is indeed the dimension of the determinantal locus. It turns out that the upper bound of $\dim W(\underline{b}; \underline{a})$, given in Theorem 4.2.7, is sharp in a number of instances. More precisely, for $2 \leq c \leq 3$, this is known (see [56], [25]), for $4 \leq c \leq 5$, it is a consequence of one of the main theorems of this section (see Corollaries 4.2.26 and 4.2.30), while for $c \geq 6$, we get the expected dimension formula for $W(\underline{b}; \underline{a})$ under more restrictive numerical assumptions (see Corollary 4.2.31). Finally, we study when the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ and when $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is generically smooth along $W(\underline{b};\underline{a})$. In Theorem 4.2.35, we show that the closure of $W(\underline{b};\underline{a})$ is a generically smooth irreducible component provided the zero-degree pieces of certain Ext^1-groups vanish. The conditions of Theorem 4.2.35 can be shown to be satisfied in a wide number of cases which we make explicit. In particular, we show that the mentioned Ext^1-groups vanish if $3 \le c \le 4$ (Corollary 4.2.37). Similarly, in Corollaries 4.2.41, 4.2.43, and 4.2.44 and as a consequence of Theorem 4.2.35, we prove that under certain numerical assumptions the closure of $W(\underline{b};\underline{a})$ is indeed a generically smooth, irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ of the expected dimension. In Examples 4.2.40 and 4.2.42, we show that this is not always the case, although the examples created are somewhat special because all the entries of the associated matrix are linear entries. We end Chapter 4 with two conjectures raised by these results and proved in many cases (see Conjectures 4.2.47 and 4.2.48).

Throughout this book, we have mentioned various open problems. Some of them and further problems related to determinantal ideals and symmetric determinantal ideals are collected in the last chapter of the book. In fact, in Chapter 5 we address for determinantal ideals and symmetric determinantal ideals the problems addressed in the previous chapters for standard determinantal ideals; namely, the G-liaison class, the multiplicity conjecture, and the unobstructedness of determinantal ideals and symmetric determinantal ideals. We collect what is known, some open problems that naturally arise in this general setup, and we add some conjectures raised in the work.

More precisely, in Section 5.1, we determine the G-liaison class of symmetric determinantal subschemes (see Proposition 5.1.11 and Theorem 5.1.12) and we also sketch the proof of Gorla's theorem: Any determinantal subscheme is glicci (see Theorem 5.1.4). We devote Section 5.2 to prove the multiplicity conjecture for symmetric determinantal ideals of codimension $\binom{m-t+2}{2}$ defined by the $t \times t$ minors of an $m \times m$ homogeneous symmetric matrix for any $t = 1, m-1$, and m, and we left open the cases $2 \le t \le m-2$. In Section 5.2, we also show that the ith total Betti number $\beta_i(R/I)$ of a symmetric determinantal ideal, defined by the submaximal minors of a homogeneous symmetric matrix, is bounded above by a function of the maximal shifts $M_i(I)$ in the minimal graded free R-resolution of R/I as well as bounded below by a function of both the maximal $M_i(I)$ and the minimal shifts $m_i(I)$. In the last section of this work, we write down a lower bound for $\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$, where $X \subset \mathbb{P}^n$ is a symmetric determinantal subscheme of codimension 3 defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix (see Theorem 5.3.5), and we analyze when the mentioned bound is sharp. This last result is a nice contribution to the classification problem of codimension r Cohen–Macaulay quotients of the polynomial ring $K[x_0, x_1, \dots, x_n]$. There is, in our opinion, little hope of solving the above classification problem in full generality and for arbitrary codimension, and in this last section we have restricted our attention to codimension 3 arithmetically Cohen–Macaulay subschemes $X \subset \mathbb{P}^n$ with homogeneous ideal generated by the submaximal minors of an $m \times m$ homogeneous symmetric matrix.

We have tried hard to keep the text as self-contained as possible. The basics of algebraic geometry supplied by Hartshorne's book [39] suffices as a foundation for this text. Some familiarity with commutative algebra, as developed in Matsumura's book [64] and Bruns–Herzog's book [9], is helpful and the rudiments on Ext and Tor contained in every introduction to homological algebra will be used freely.

Acknowledgments. This book grew out of a series of lectures that the author will deliver at a winter school at the Indian Institute of Technology, Mumbai, January 2008 (see [70]). The author is grateful to the main organizers of this School/Conference, Luchezar Avramov, Maria Evelina Rossi, Marc Chardin, Tony Joseph Puthenpurakal, and Jugal Verma for giving her the opportunity to speak about one of her favorite subjects, determinantal ideals.

The author is also grateful to Joan Elias, Laura Costa, and José M. Giral. They read parts of earlier versions of this text and pointed out many things that needed fixing.

The author is greatly indebted to her coauthors of [56] for the enjoyable collaboration which led to most of the material described in Chapter 2 of this work: they are Jan O. Kleppe, Juan C. Migliore, Uwe Nagel, and Chris Peterson. Last but not least, the author thanks Jan O. Kleppe as parts of Chapters 4 and 5 grew out of a long time cooperation.

Chapter 1

Background

Throughout this book, $\mathbb{P}^n$ will be the n-dimensional projective space over an algebraically closed field K of characteristic zero, $R = K[x_0, x_1, \dots, x_n]$ and $\mathfrak{m} = (x_0, x_1, \dots, x_n)$ its homogeneous maximal ideal. If M is a graded R-module, we distinguish two types of duals of M: the R-dual $M^* := \operatorname{Hom}_R(M, R)$ and the K-dual $M^\vee := \operatorname{Hom}_K(M, K)$. A *scheme* $V \subset \mathbb{P}^n$ will mean an equidimensional locally Cohen–Macaulay closed subscheme of $\mathbb{P}^n$. For a subscheme V of $\mathbb{P}^n$ we denote by $\mathcal{I}_V$ its ideal sheaf, $I(V) = H^0_*(\mathcal{I}_V) := \bigoplus_{t \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{I}_V(t))$ its saturated homogeneous ideal (unless $V = \emptyset$, in which case we let $I(V) = \mathfrak{m}$), $A(V) = R/I(V)$ the homogeneous coordinate ring, and $\mathcal{N}_V = \mathcal{H}om(\mathcal{I}_V, \mathcal{O}_V)$ the normal sheaf of V.

As we pointed out in the introduction, the main purpose of this first chapter is to give some of the definitions and the necessary background for the material in the subsequent chapters. The topics of Section 1.1 are, minimal free resolutions, arithmetically Cohen–Macaulay schemes, and arithmetically Gorenstein schemes. In Section 1.2, we provide the basic facts on determinantal ideals and symmetric determinantal ideals as well as the associated complexes. Finally, in Section 1.3, we overview the known results on liaison theory needed in the sequel.

1.1 Minimal free resolutions, ACM schemes, and AG schemes

Minimal free resolutions

Let $R = K[x_0, \dots, x_n]$ be the polynomial ring. By our standard conventions, a homomorphism $\varphi : M \longrightarrow N$ of graded R-modules is graded of degree zero, i.e., $\varphi([M]_j) \subset [N]_j$ for all $j \in \mathbb{Z}$.

Definition 1.1.1. Let M be a graded R-module. Then $N \neq 0$ is said to be a *k-syzygy* of M (as R-module) if there is an exact sequence of graded R-modules

$$0 \longrightarrow N \longrightarrow F_k \xrightarrow{\varphi_k} F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} M \longrightarrow 0, \tag{1.1}$$

where the modules F_i are free R-modules. A module is called a *k-syzygy* if it is a k-syzygy of some module.

Note that a $(k+1)$-syzygy is also a k-syzygy. Moreover, every k-syzygy N is a maximal R-module, i.e., $\dim N = \dim R$.

If M is a finitely generated graded R-module, let $\operatorname{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal $J \subset R$ and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$. Let $H^i_J(-)$ be the ith right derived functor of the functor $\Gamma_J(-)$ of sections with support in $\operatorname{Spec}(R/J)$ defined by

$$\Gamma_J(M) := \{m \in M \mid \operatorname{Supp} Rm \subset V(J)\}.$$

We define the *ith local cohomology functor* $H^i_{\mathfrak{m}}(-)$ as the ith right derived functor of the left exact functor $H^0_{\mathfrak{m}}(-)$ defined by

$$H^0_{\mathfrak{m}}(M) := \{m \in M \mid \mathfrak{m}^k \cdot m = 0 \text{ for some } k \in \mathbb{N}\}.$$

The Krull dimension and the depth of a finitely generated graded R-module M are cohomologically characterized by

$$\dim M = \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}, \tag{1.2}$$
$$\operatorname{depth} M = \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}. \tag{1.3}$$

Cutting the long exact sequence (1.1) into short exact sequences, we easily obtain the following Lemma.

Lemma 1.1.2. *If N is a k-syzygy of an R-module M, then*

$$H^i_{\mathfrak{m}}(N) \cong H^{i-k}_{\mathfrak{m}}(M) \quad \textit{for all} \quad i < \dim R.$$

It follows that the depth of a k-syzygy is al least k.

Now we relate the local cohomology with the sheaf cohomology. Let $\mathcal{F}$ be a sheaf of modules over $X = \operatorname{Proj}(A)$ where A is a graded K-algebra. Its cohomology modules are denoted by

$$H^i_*(X, \mathcal{F}) = \oplus_{j \in \mathbb{Z}} H^i(X, \mathcal{F}(j)).$$

There are two functors relating graded A-modules M and sheaves of modules over X: the sheafification functor which associates with each graded A-module M the sheaf $\tilde{M}$. This functor is exact. In the opposite direction, there is the "twisted global sections" functor which associates with each sheaf of modules $\mathcal{F}$ over X the graded A-module $H^0_*(X,\mathcal{F})$. This functor is only left exact. If $\mathcal{F}$ is a quasi-coherent sheaf, then the sheaf $\widetilde{H^0_*(X,\mathcal{F})}$ is canonically isomorphic to $\mathcal{F}$. However, if M is a graded A-module, then the module $H^0_*(X,\tilde{M})$ is not isomorphic to M in general. Thus these two functors do not establish an equivalence of categories between the category of graded A-modules and the category of quasi-coherent sheaves of modules over X. However, there is the following comparison result (cf. [84]).

Proposition 1.1.3. *Let M be a graded A-module. Then there is an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow H^0_*(X,\tilde{M}) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0$$

and, for all $i \geq 1$, there are isomorphisms

$$H^{i+1}_{\mathfrak{m}}(M) \cong H^i_*(X,\tilde{M}).$$

Corollary 1.1.4. *Let $X \subset \mathbb{P}^n = \mathrm{Proj}(R)$ be a closed subscheme of dimension $d \leq n-1$. Then there are graded isomorphisms*

$$H^i_*(\mathbb{P}^n,\mathcal{I}_X) \cong H^i_{\mathfrak{m}}(R/I(X)) \quad \textit{for all} \quad i = 1,\dots,d+1.$$

Proof. Since $H^i_{\mathfrak{m}}(R) = 0$ if $i \leq n$, the cohomology sequence of

$$0 \longrightarrow I(X) \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0,$$

together with the last proposition, gives us

$$H^i_{\mathfrak{m}}(R/I(X)) \cong H^{i+1}_{\mathfrak{m}}(I(X)) \cong H^i_*(\mathbb{P}^n,\mathcal{I}_X) \quad \text{for all} \quad i < n. \qquad \square$$

Definition 1.1.5. Let $\varphi : F \longrightarrow M$ be a morphism of R-modules where F is free. Then φ is said to be a *minimal homomorphism* if $\varphi \otimes id_{R/\mathfrak{m}} : F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M$ is the zero map in case M is free and an isomorphism in case φ is surjective.

In the situation of the definition above, N is said to be a *minimal k-syzygy of M* if the morphisms φ_i, $i = 1,\dots,k$, of the resolution (1.1) are minimal morphisms. If, in addition, N is a free R-module, then the exact sequence (1.1) is called a *minimal free resolution* of M.

Nakayama's lemma implies easily that minimal k-syzygies of M are unique up to isomorphism and that the minimal free resolution of M is unique up to isomorphism of complexes.

Remark 1.1.6. Let

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

be a free R-resolution of M. Then it is minimal if and only if (after choosing bases for $F_0,\dots,F_s$) the matrices representing $\varphi_1,\dots,\varphi_s$ have all the entries in the maximal ideal $\mathfrak{m}$ of R.

Definition 1.1.7. Let M be a graded R-module. M is said to be of *finite projective dimension* if there is a free R-resolution of M,

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0.$$

The minimum of the length s of such free resolution is called the *projective dimension of M* and it is denoted by $\mathrm{pd}(M)$.

Let M be a finitely generated graded R-module, the Hilbert syzygy theorem says that the projective dimension of M, $\mathrm{pd}(M)$, is less than or equal to $n+1$. In fact, we have the following.

Theorem 1.1.8 (Hilbert syzygy theorem). *Let M be a finitely generated graded R-module and let*

$$\begin{aligned} 0 \longrightarrow E \longrightarrow \oplus_{1\le j\le a_n} R(-d_{n,j}) \longrightarrow \cdots \longrightarrow \oplus_{1\le j\le a_1} R(-d_{1,j}) \\ \longrightarrow \oplus_{1\le j\le a_0} R(-d_{0,j}) \longrightarrow M \longrightarrow 0 \end{aligned}$$

be an exact sequence. Then there exist integers a_{n+1} and $d_{n+1,j}$ with $1 \le j \le a_{n+1}$ such that $E = \oplus_{1\le j\le a_{n+1}} R(-d_{n+1,j})$, i.e., $\mathrm{pd}(M) \le n+1$.

Proof. We proceed by induction on the number $n+1$ of variables. If $n+1 = 0$, then $R = K$ and M is a K-vector space of finite dimension. Fix $(e_1, \ldots, e_m)$ a homogeneous basis of M and $d_i = \deg(e_i)$. So, the morphism

$$\sum_{1\le j\le m} f_i : \oplus_{1\le j\le m} K(-d_i) \longrightarrow M$$

defined by $f_i(1) = e_i$ is an isomorphism of graded K-modules.

Assume $n+1 > 0$. Set $N = \mathrm{Ker}(\oplus_{1\le j\le a_0} R(-d_{0,j}) \longrightarrow M)$ and consider the following commutative diagram, with exact rows and columns:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & E(-1) & \longrightarrow & \oplus_{j=1}^{a_n} R(-d_{n,j}-1) & \to \cdots \to & \oplus_{j=1}^{a_1} R(-d_{1,j}-1) & \to & N(-1) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & E & \longrightarrow & \oplus_{j=1}^{a_n} R(-d_{n,j}) & \to \cdots \to & \oplus_{j=1}^{a_1} R(-d_{1,j}) & \to & N & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & E/x_nE & \longrightarrow & \oplus_{j=1}^{a_n} R'(-d_{n,j}) & \to \cdots \to & \oplus_{j=1}^{a_1} R'(-d_{1,j}) & \to & N/x_nN & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

where $R' = R/x_nR \cong K[x_0, x_1, \ldots, x_{n-1}]$.

By hypothesis of induction, there exist integers m and ℓ_i, $1 \le i \le m$, such that $E/x_nE \cong \oplus_{1\le i\le m} R'(-\ell_i)$. Let us check that $E \cong \oplus_{1\le i\le m} R(-\ell_i)$. To this end, we choose homogeneous elements $z_1, z_2, \ldots, z_m \in E$ of degree $\deg(z_i) = \ell_i$ such that $\overline{z_1}, \overline{z_2}, \ldots, \overline{z_m}$ is a basis of E/x_nE. We denote by E' the graded submodule of E generated by $z_1, z_2, \ldots, z_m$. We have $E = E' + x_nE$ and applying Nakayama's lemma, we conclude that $E = E'$. It remains to see that there are no nontrivial relations among $z_1, z_2, \ldots, z_m$.

Let $\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_m z_m = 0$ be a homogeneous nontrivial relation of minimal degree. Since $\overline{\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_m z_m} = 0$, we have $\alpha_i \in x_n R$ for all $1 \leq i \leq m$. Thus, $\alpha_i = x_n \beta_i$ and $x_n(\beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_m z_m) = 0$. But x_n is a non-zero divisor of E and we conclude that $\beta_1 z_1 + \beta_2 z_2 + \cdots + \beta_m z_m = 0$ which contradicts the fact that $\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_m z_m = 0$ is a relation of minimal degree unless $\beta_i = 0$ and $\alpha_i = 0$ for all $1 \leq i \leq m$. We have proved that $(z_1, \ldots, z_m)$ is a basis of E and hence $E \cong \oplus_{1 \leq i \leq m} R(-\ell_i)$, which proves what we want. □

Remark 1.1.9. The bound given in Theorem 1.1.8 is sharp. More precisely, set $R = K[x_0, \ldots, x_n]$. For any integer i, $0 \leq i \leq n+1$, there exists a finitely generated graded R-module M such that $\mathrm{pd}(M) = i$. Indeed, if $i = 0$ we take $M = R$ and if $1 \leq i \leq n+1$ we take $M = (x_0, \ldots, x_{i-1})$.

Definition 1.1.10. Let M be a graded R-module. If

$$\begin{aligned} 0 \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \longrightarrow \cdots \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(M)} \\ \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0 \end{aligned}$$

is a minimal graded free R-resolution of M, the integer $\beta_{i,j}(M) = \dim \mathrm{Tor}_i^R(M, K)_j$ is called the (i,j)th *graded Betti number* of M and the integer $\beta_i := \sum_{j \in \mathbb{Z}} \beta_{i,j}(M)$ is called the ith *total Betti number* of M.

As we will see in this book, many important numerical invariants of a homogeneous ideal $I \subset R$ and the associated scheme $X = \mathrm{Proj}(R/I)$ can be read off from the minimal graded free R-resolution of R/I. For instance, the Castelnuovo–Mumford regularity, the Hilbert polynomial, and hence the multiplicity $e(R/I)$ of I, can be written down in terms of the shifts j such that $\beta_{i,j}(R/I) \neq 0$ for some i, $1 \leq i \leq n+1$.

We briefly recall the definition of Castelnuovo–Mumford regularity of a graded R-module which is an important measure of how complicated the module is.

Definition 1.1.11. Let M be a graded R-module and let

$$\begin{aligned} 0 \longrightarrow \oplus_{j=1}^{r_p} R(-a_{p,j}) \longrightarrow \oplus_{j=1}^{r_{p-1}} R(-a_{p-1,j}) \cdots \longrightarrow \oplus_{j=1}^{r_1} R(-a_{1,j}) \\ \longrightarrow \oplus_{j=0}^{r_0} R(-a_{0,j}) \longrightarrow M \longrightarrow 0 \end{aligned}$$

be its minimal free R-resolution. Then its *Castelnuovo–Mumford regularity* is the number

$$\mathrm{reg}(M) := \max_{i,j}(a_{i,j} - i).$$

If $X \subset \mathbb{P}^n$ is a subscheme, we define the *Castelnuovo–Mumford regularity* of X as the Castelnuovo–Mumford regularity of its homogeneous ideal $I(X)$.

We have the following important characterization of the regularity:

$$\operatorname{reg}(M) = \max_{i\in\mathbb{Z}}\{i + a(H^i_{\mathfrak{m}}(M))\},$$

where for any nonzero finitely generated graded R-module, we set

$$a(N) = sup\{t \in \mathbb{Z} \mid N_t \neq 0\}.$$

If $N = 0$, we set $a(N) = -\infty$. In particular, if M is a nonzero graded R-module of finite length, then

$$\operatorname{reg}(M) = \max\{d \mid M_d \neq 0\}.$$

Let $I \subset R$ be a homogeneous ideal. The Castelnuovo–Mumford regularity of R/I contains quite a lot of geometric information as we will illustrate in the following example.

Example 1.1.12. Let $I \subset R = K[x_0, x_1, x_2]$ be the homogeneous ideal of 3 collinear points in the plane $\mathbb{P}^2$ and let $J \subset R$ be the homogeneous ideal of 3 noncollinear points. The minimal free R-resolutions of R/I and R/J are of the following types:

$$0 \longrightarrow R(-4) \longrightarrow R(-1) \oplus R(-3) \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

and

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R/J \longrightarrow 0.$$

Therefore, $\operatorname{reg}(R/J) = 1$ and $\operatorname{reg}(R/I) = 2$. Hence, the Castelnuovo–Mumford regularity tells us exactly whether or not 3 points are collinear.

Arithmetically Cohen–Macaulay schemes

We now recall the definition of Cohen–Macaulay module and that of arithmetically Cohen–Macaulay subscheme of $\mathbb{P}^n$.

Definition 1.1.13. A finitely generated R-module M is said to be a *Cohen–Macaulay module* if $\operatorname{depth} M = \dim M$ or $M = 0$.

The following corollary immediately follows from (1.2) and (1.3).

Corollary 1.1.14. *A finitely generated R-module M is Cohen–Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim M$.*

Definition 1.1.15. A subscheme $X \subset \mathbb{P}^n$ is said to be arithmetically Cohen–Macaulay (briefly, ACM) if its homogeneous coordinate ring $R/I(X)$ is a Cohen–Macaulay ring; i.e., $\operatorname{depth}(R/I(X)) = \dim(R/I(X))$.

Thanks to the graded version of the Auslander–Buchsbaum formula (for any finitely generated R-module M),

$$\operatorname{pd}(M) = n + 1 - \operatorname{depth}(M),$$

we deduce that a subscheme $X \subset \mathbb{P}^n$ is ACM if and only if the projective dimension of $R/I(X)$ is equal to the codimension of X; i.e.,

$$\operatorname{pd}(R/I(X)) = \operatorname{codim} X. \tag{1.4}$$

Hence, if $X \subset \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of $I(X)$ is of the form

$$0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \xrightarrow{\varphi_c} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} I(X) \longrightarrow 0,$$

where $F_i = \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}}$, $i = 1,\dots,c$ (in this setting, minimal means that $\operatorname{im}\varphi_i \subset \mathfrak{m}F_{i-1}$).

Definition 1.1.16. If $X \subset \mathbb{P}^n$ is an ACM subscheme, then the rank of the last free R-module in a minimal free R-resolution of $I(X)$ (or, equivalently of $R/I(X)$) is called the *Cohen–Macaulay type* of X.

It is easy to check that if $X \subset \mathbb{P}^n$ is an ACM subscheme of dimension ≥ 1 and F is a general polynomial of degree d cutting out on X a scheme $Y \subset X \subset \mathbb{P}^n$, then Y is also ACM and the Cohen–Macaulay type of Y is the same as that of X.

Definition 1.1.17. Given a closed subscheme $X \subset \mathbb{P}^n$ of dimension $d \geq 1$, we define the *deficiency modules* of X, $M^i(X)$, $i = 1,\dots,d$, as the ith cohomology module of the ideal sheaf of X,

$$M^i(X) = H^i_*\mathcal{I}_X = \oplus_t H^i(\mathbb{P}^n, \mathcal{I}_X(t)) \quad \text{for } 1 \leq i \leq d.$$

The deficiency modules can also be written in terms of Ext,

$$M^i(X) = (\operatorname{Ext}_R^{n-i+1}(R/I(X), R)(-n-1))^\vee \quad \text{for } 1 \leq i \leq d.$$

When X is a curve, its deficiency module $M^1(X)$ is also called *Rao module* or *Hartshorne-Rao module.*

As we will see later, these modules play an important role in liaison theory mainly because (up to shift and dual) they are invariant under both CI-liaison and G-liaison (see [77]). Furthermore, the deficiency modules measure the failure of a subscheme $X \subset \mathbb{P}^n$ to be ACM. In fact, we have that any 0-dimensional subscheme $X \subset \mathbb{P}^n$ is an ACM scheme and, from the characterization of the deficiency modules in terms of Ext, it is easy to deduce that if $\dim X = d \geq 1$, then X is ACM if and only if $M^i(X) = 0$ for $1 \leq i \leq d$.

Example 1.1.18. (a) *Complete intersections.* If X is a subscheme in $\mathbb{P}^n$ of codimension r, then the number of generators of $I(X)$ is at least r, and we say that X is a *complete intersection* if the number of generators of $I(X)$ is equal to r, the codimension of X. From the algebraic point of view, $X \subset \mathbb{P}^n$ is a complete intersection subscheme if and only if $I(X) = (F_1,\dots,F_r)$, where $(F_1,\dots,F_r)$ is a regular sequence. Moreover, the minimal free R-resolution of $R/I(X)$ is known as the Koszul resolution and it is given by

$$0 \longrightarrow R\left(-\sum_{i=1}^{r} d_i\right) \longrightarrow \cdots \longrightarrow \wedge^2(\oplus_{1\leq i\leq r}R(-d_i))$$
$$\longrightarrow \oplus_{1\leq i\leq r}R(-d_i) \longrightarrow I(X) \longrightarrow 0$$

where $d_i = \deg(F_i)$. Therefore, the projective dimension of $R/I(X)$ is equal to the codimension r. So, complete intersection schemes are the simplest examples of ACM subschemes $X \subset \mathbb{P}^n$ of Cohen–Macaulay type 1 and arbitrary codimension.

From now on, we will say that a subscheme $X \subset \mathbb{P}^n$ is a *complete intersection of type* $(d_1,\dots,d_r)$ if it is a complete intersection of codimension r and $I(X)$ is generated by homogeneous forms F_i of degree d_i, $1 \leq i \leq r$.

(b) As an example of ACM subscheme $X \subset \mathbb{P}^n$ which is not a complete intersection subscheme, we have the twisted cubic curve $C \subset \mathbb{P}^3$ with $I(C) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2)$. Its minimal free $R = K[x_0,x_1,x_2,x_3]$-resolution looks like (Hilbert–Burch's theorem)

$$0 \longrightarrow R(-3)^2 \xrightarrow{B} R(-2)^3 \xrightarrow{A} I(C) \longrightarrow 0,$$

where

$$B = \begin{pmatrix} -x_1 & x_0 \\ x_2 & -x_1 \\ -x_3 & x_2 \end{pmatrix},$$
$$A = \begin{pmatrix} x_0x_2 - x_1^2 & x_0x_3 - x_1x_2 & x_1x_3 - x_2^2 \end{pmatrix}.$$

So the twisted cubic is an ACM curve with Cohen–Macaulay type 2 and it is not a complete intersection.

(c) The smooth rational quartic curve $X \subset \mathbb{P}^3$ given by $I(X) = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3, x_1x_3^2 - x_2^3, x_2x_0^2 - x_1^3)$ is an example of non-ACM subscheme. Indeed, $I(X)$ has the following minimal free $R = K[x_0,x_1,x_2,x_3]$-resolution:

$$0 \longrightarrow R(-5) \xrightarrow{C} R(-4)^4 \xrightarrow{B} R(-3)^3 \oplus R(-2) \xrightarrow{A} I(X) \longrightarrow 0$$

where

$$A = \begin{pmatrix} x_0x_3 - x_1x_2 & x_2x_0^2 - x_1^3 & x_1x_3^2 - x_2^3 & x_0x_2^2 - x_1^2x_3 \end{pmatrix},$$
$$B = \begin{pmatrix} -x_2^2 & x_1x_3 & -x_0x_2 & -x_1^2 \\ 0 & 0 & x_3 & x_2 \\ x_1 & -x_0 & 0 & 0 \\ x_3 & -x_2 & -x_1 & -x_0 \end{pmatrix},$$
$$C = \begin{pmatrix} -x_0 \\ -x_1 \\ x_2 \\ -x_3 \end{pmatrix}.$$

So, X is not ACM by (1.4).

(d) An important example of ACM subschemes of $\mathbb{P}^n$ are the so-called arithmetically Gorenstein subschemes (see Definition 1.1.26) which play an important role in this memoir.

(e) 0-*dimensional schemes.* One always has $\operatorname{depth}(R/I(X)) \leq \dim(R/I(X))$ for any subscheme $X \subset \mathbb{P}^n$. Therefore, any 0-dimensional subscheme $X \subset \mathbb{P}^n$ is automatically an ACM subscheme.

For any finitely generated graded R-module M we denote by h_M its *Hilbert function* defined by $h_M(t) = \dim_k [M]_t$. For large values of t, this function is a polynomial, p_M, of degree $\dim(M) - 1$ called the *Hilbert polynomial* of M.

Definition 1.1.19. Let M be a finitely generated graded R-module, let d be its Krull dimension, and let $p_M(x) = a_0 x^{d-1} + a_1 x^{d-2} + \cdots + a_{d-1}$ be its Hilbert polynomial. We define the *multiplicity of M*, $e(M) := a_0(d-1)!$. So, we have

$$p_M(x) = \frac{e(M)}{(d-1)!} x^{d-1} + \cdots .$$

The Hilbert function, h_X, and the Hilbert polynomial, p_X, of a subscheme $X \subset \mathbb{P}^n$ are the corresponding functions of its homogeneous coordinate ring $R/I(X)$.

It is worth pointing out that given a graded R-module M, the graded Betti numbers $\beta_{ij}(M)$ of M determine the Hilbert function h_M of M, and this in turn determines the Hilbert polynomial p_M of M. As we might expect, the finer the information the more difficult it is to obtain; in particular, while the Hilbert polynomial of most varieties is relatively accessible, there are relatively few varieties whose graded Betti numbers can be calculated. By way of an example, the Hilbert polynomial of an arbitrary collection of d points in $\mathbb{P}^n$ is trivial; the Hilbert function is easy to determine for d general points and quite difficult in general; we do not know the graded Betti numbers of a collection of d general points in $\mathbb{P}^n$ for all d and n. To summarize, finding the resolution (or, equivalently, the graded Betti numbers) of the ideal of a variety X is not an effective way to find things out about X, except in the realm of machine calculation; for the most part it is more a source of interesting questions than of answers.

Example 1.1.20 (Hilbert function of the rational normal curve). Let $X \subset \mathbb{P}^d$ be a rational normal curve. X is defined as the image of the map

$$v_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$
$$[a : b] \mapsto [a^d : a^{d-1}b : \cdots : ab^{d-1} : b^d].$$

Under the map v_d, the homogeneous polynomials of degree m in the coordinates $[z_0 : \cdots : z_d]$ on $\mathbb{P}^d$ pull back to give all homogeneous polynomials of degree $m \cdot d$ in the coordinates x_0 and x_1 of $\mathbb{P}^1$. Therefore, $(R/I(X))_m \cong K[x_0, x_1]_{m \cdot d}$. So, for all $m \geq 0$, we have

$$h_X(m) = p_X(m) = d{\cdot}m + 1.$$

Example 1.1.21 (Hilbert function of the Veronese variety). Let $X \subset \mathbb{P}^N$ be the Veronese variety defined as the image of the map

$$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N,$$
$$[x_0 : x_1 : \cdots : x_n] \mapsto [\cdots : x^I : \cdots],$$

where x^I ranges over all monomials of degree d in $x_0, \ldots, x_n$. Hence, $N = \binom{n+d}{d} - 1$. Under the map v_d, the homogeneous polynomials of degree m on $\mathbb{P}^N$ pull back to give all homogeneous polynomials of degree $m{\cdot}d$ on $\mathbb{P}^n$. Therefore, we have $(K[x_0, x_1, \ldots, x_N]/I(X))_m \cong K[x_0, x_1, \ldots, x_n]_{m\cdot d}$. So, for all $m \geq 0$, we have

$$h_X(m) = p_X(m) = \binom{m{\cdot}d + n}{n}.$$

The following result relates the depth and the Castelnuovo–Mumford regularity of a finitely generated graded Cohen–Macaulay R-module M with the smallest integer such that the Hilbert polynomial and the Hilbert function of M coincide.

Proposition 1.1.22. *Let M be a finitely generated graded Cohen–Macaulay R-module. If s is the smallest integer such that $h_M(d) = p_M(d)$ for all $d \geq s$, then*

$$s = 1 - \operatorname{depth}(M) + \operatorname{reg}(M).$$

Proof. See [23, Corollary 4.8]. □

For a numerical function $h : \mathbb{Z} \longrightarrow \mathbb{Z}$ we define its first difference by $\Delta h(j) = h(j) - h(j-1)$ and the higher differences by $\Delta^i = \Delta(\Delta^{i-1} h)$ and $\Delta^0 h = h$.

Remark 1.1.23. Suppose that $X \subset \mathbb{P}^n$ is an ACM scheme of dimension d. Let $\ell_1, \ldots, \ell_{d+1} \in R$ be linear forms such that $R/(I(X) + (\ell_1, \ldots, \ell_{d+1}))$ has dimension zero. Then $R/(I(X) + (\ell_1, \ldots, \ell_{d+1}))$ is called an *Artinian reduction* of $R/I(X)$. For its Hilbert function we have

$$h_{R/(I(X)+(\ell_1,\ldots,\ell_{d+1}))}(t) = \Delta^{d+1} h_{R/I(X)}(t).$$

Since $R/(I(X) + (\ell_1, \ldots, \ell_{d+1}))$ is finite dimensional as a K-vector space, we have that $h_{R/(I(X)+(\ell_1,\ldots,\ell_{d+1}))}(t)$ is a finite sequence of nonzero integers,

$$1 \; c \; h_2 \; h_3 \ldots \; h_s \; 0 \ldots \; .$$

The above sequence is called the *h-vector* of X. In particular, c is the *embedding codimension* of X; i.e., c is the codimension of X inside the smallest linear space containing X. Moreover, the Hilbert function of X can be recovered from the h-vector by "integrating".

Example 1.1.24 (h-vector of the rational normal curve). Let $X \subset \mathbb{P}^d$ be a rational normal curve defined as the image of the map

$$v_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$
$$[a : b] \mapsto [a^d : a^{d-1}b : \cdots : ab^{d-1} : b^d].$$

By Example 1.1.20, for all $m \geq 0$, we have

$$h_X(m) = p_X(m) = d{\cdot}m + 1.$$

So, the h-vector of X is $1 \; d-1 \; 0 \; 0 \; \ldots .$

If $X \subset \mathbb{P}^n$ is a closed subscheme of codimension c and $A = R/I(X)$, we denote by K_X (or sometimes K_A) its *canonical module*

$$K_X = \operatorname{Ext}^c_R(R/I(X), R)(-n-1),$$

and by ω_X its *canonical sheaf*

$$\omega_X = \mathcal{E}xt^c_R(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n})(-n-1) \ .$$

Remark 1.1.25. An important fact is that if $X \subset \mathbb{P}^n$ is an ACM scheme, then the dual of the resolution of $R/I(X)$ is again a resolution, this time for the twist of the canonical module K_X of X. In fact, the resolution

$$0 \longrightarrow F_{n-d} \longrightarrow F_{n-d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0$$

yields the resolution

$$0 \longrightarrow R \longrightarrow F_1^\vee \longrightarrow \cdots \longrightarrow F_{n-d-1}^\vee \longrightarrow F_{n-d}^\vee$$
$$\longrightarrow \operatorname{Ext}^{n-d}_R(R/I(X), R) = K_X(n+1) \longrightarrow 0.$$

To see this, we use the fact that $\operatorname{Ext}^i_R(R/I(X), R) = 0$ for $i < \operatorname{codim}(X)$. In particular, the Cohen–Macaulay type of X coincides with the minimal number the of generators of its canonical module, and if X has Cohen–Macaulay type 1 (i.e., X is arithmetically Gorenstein), then the canonical module is cyclic.

Arithmetically Gorenstein schemes

Definition 1.1.26. A codimension c subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* (briefly AG), if its homogeneous coordinate ring $R/I(X)$ is a Gorenstein ring or, equivalently, its saturated homogeneous ideal, $I(X)$, has a minimal free graded R-resolution of the following type:

$$0 \longrightarrow R(-t) \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(-n_{c-1,i}) \longrightarrow \cdots \longrightarrow \oplus_{i=1}^{\alpha_1} R(-n_{1,i}) \longrightarrow I(X) \longrightarrow 0.$$

In other words, an AG scheme is an ACM scheme with Cohen–Macaulay type 1.

By Remark 1.1.25 the canonical module K_X of an AG subscheme $X \subset \mathbb{P}^n$ is cyclic. In fact, the following proposition is true.

Proposition 1.1.27. *Let X be an ACM subscheme of $\mathbb{P}^n$ of codimension c. Then the following conditions are equivalent:*

(1) *X has Cohen–Macaulay type 1 (i.e., X is AG);*
(2) *$R/I(X) \cong K_X(t)$ for some $t \in \mathbb{Z}$;*
(3) *the minimal free R-resolution of $R/I(X)$ is self-dual, up to twisting by $n+1$.*

Proof. (1) implies (2). We consider a minimal free graded R-resolution of $R/I(X)$,

$$\begin{aligned} 0 \longrightarrow R(-t) \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(-n_{c-1,i}) \longrightarrow \cdots \\ \longrightarrow \oplus_{i=1}^{\alpha_1} R(-n_{1,i}) \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0. \end{aligned}$$

Dualizing it we get (see Remark 1.1.25)

$$\begin{aligned} 0 \longrightarrow R \longrightarrow \oplus_{i=1}^{\alpha_1} R(n_{1,i}) \longrightarrow \cdots \longrightarrow \oplus_{i=1}^{\alpha_{c-1}} R(n_{c-1,i}) \longrightarrow R(t) \\ \longrightarrow \operatorname{Ext}_R^c(R/I(X), R) = K_X(n+1) \longrightarrow 0. \end{aligned}$$

Therefore, $K_X \cong R/J(\ell)$ for some ideal J and some integer ℓ. On the other hand, since X is ACM and $\operatorname{Ext}_R^c(R/I(X), R) = K_X(n+1)$, we have that $I(X) = \operatorname{Ann}_R(K_X)$. Combining these facts we get that $J = I(X)$ which proves (2).

(2) implies (3). This follows from Remark 1.1.25.

(3) implies (1). The fact that the minimal free R-resolution of $R/I(X)$ is self-dual implies that the Cohen–Macaulay type of X is 1, which proves (1). ☐

In general, it is a difficult matter to construct AG subschemes $X \subset \mathbb{P}^n$, apart from complete intersections, with prescribed properties (e.g., with given Hilbert function or containing a given scheme).

Example 1.1.28. (a) The Koszul resolution of a complete intersection tells us that any complete intersection subscheme $X \subset \mathbb{P}^n$ is an AG subscheme.

(b) In codimension 2, the converse of (a) holds. If X is AG, then from the minimal free R-resolution

$$0 \longrightarrow R(-t) \longrightarrow F_1 \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0,$$

one quickly sees that the rank of F_1 is 2; i.e., X is a complete intersection. So, in codimension 2 AG subschemes and complete intersection subschemes coincide.

(c) In higher codimension, any complete intersection subscheme is AG but not vice versa. Indeed, a set X of $n+2$ points in $\mathbb{P}^n$ in linear general position (i.e., no $n+1$ on a hyperplane) is AG (this easily follows from the fact that a zero scheme $X \subset \mathbb{P}^n$ is AG if and only if it has symmetric h-vector and it has the Cayley–Bacharach property; see Definition 1.1.29), but X is not a complete intersection. In particular, a set X of 5 general points in $\mathbb{P}^3$ is ACM (because it is 0-dimensional) and its homogeneous coordinate ring $R/I(X)$ has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0.$$

So, X is AG (the Cohen–Macaulay type is 1) and X is not a complete intersection because it has codimension 3 and $I(X)$ is generated by 5 quadrics. So, in higher codimension it is no longer true that complete intersection subschemes and AG subschemes coincide.

(d) Let $X \subset \mathbb{P}^4$ be the elliptic quintic curve with homogeneous ideal $I(X) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_0x_4 - x_3x_1, x_1x_3 - x_2^2, x_1x_4 - x_2x_3)$. X is an AG curve. In fact, $I(X)$ has the following minimal free $R = K[x_0, x_1, x_2, x_3, x_4]$-resolution:

$$0 \longrightarrow R(-5) \xrightarrow{C} R(-3)^5 \xrightarrow{B} R(-2)^5 \xrightarrow{A} I(X) \longrightarrow 0,$$

where

$$A = \begin{pmatrix} x_0x_2 - x_1^2 & x_2x_1 - x_0x_3 & x_1x_3 - x_0x_4 & x_2^2 - x_0x_4 & x_2x_3 - x_1x_4 \end{pmatrix},$$

$$C = \begin{pmatrix} -x_2^2 + x_1x_4 \\ x_0x_2 - x_1^2 \\ -x_2x_1 + x_0x_3 \\ x_2x_3 - x_1x_4 \\ x_1x_3 - x_0x_4 \end{pmatrix},$$

$$B = \begin{pmatrix} -x_3 & 0 & x_4 & -x_2 & x_3 \\ 0 & x_4 & 0 & -x_1 & x_2 \\ -x_1 & 0 & x_2 & -x_0 & x_1 \\ 0 & -x_3 & 0 & x_0 & -x_1 \\ x_0 & x_2 & -x_1 & 0 & 0 \end{pmatrix}.$$

We easily compute the Hilbert polynomial of X and we get $p_X(t) = 5t$. So, X is an elliptic ($p_a(X) = 1$) quintic ($\deg(X) = 5$) curve. Moreover, the projective dimension of $R/I(X)$ is 3 and, hence, the elliptic quintic X is an ACM curve with Cohen–Macaulay type 1 and 5 minimal generators. Therefore, X is an AG curve but not a complete intersection curve.

Definition 1.1.29. Let $X \subset \mathbb{P}^n$ be a 0-dimensional scheme and let $r+1 = \operatorname{reg}(\mathcal{I}_X)$. Then X has the Cayley–Bacharach property if, for every subscheme $Y \subset X$ with $\deg(Y) = \deg(X) - 1$, we have $h^0(\mathbb{P}^n, \mathcal{I}_Y(r-1)) = h^0(\mathbb{P}^n, \mathcal{I}_X(r-1))$.

1.2 Determinantal ideals

The main purpose of this section is to provide the background and basic results on standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals as well as good and standard determinantal schemes needed later. We refer the reader to [10] and [22] for more details.

Definition 1.2.1. Let $\mathcal{A}$ be a $p \times q$ matrix with entries in R. We say that $\mathcal{A}$ is a *t-homogeneous matrix* if the minors of $\mathcal{A}$ of size $j \times j$ are homogeneous polynomials for all $j \leq t$. We say that $\mathcal{A}$ is a *homogeneous matrix* if its minors of any size are homogeneous.

If $\mathcal{A}$ is a homogeneous matrix, we denote by $I(\mathcal{A})$ the ideal of R generated by the maximal minors of $\mathcal{A}$. If $\mathcal{A}$ is a t-homogeneous matrix, for any $j \leq t$, we denote by $I_j(\mathcal{A})$ the ideal of R generated by the $j \times j$ minors of $\mathcal{A}$.

Remark 1.2.2. Note that associated with any $p \times q$ homogeneous matrix $\mathcal{A}$ we have a degree 0 morphism $\phi : F \longrightarrow G$ of free graded R-modules of rank p and q, respectively. We set $I(\phi) = I(\mathcal{A})$.

Definition 1.2.3. A homogeneous ideal $I \subset R$ is called a *determinantal* ideal if

(1) there exists an r-homogeneous matrix $\mathcal{A}$ of size $p \times q$ with entries in R, such that $I = I_r(\mathcal{A})$, and
(2) $ht(I) = (p - r + 1)(q - r + 1)$.

Such ideal will be often denoted $I_{p,q,r}(\mathcal{A})$ (or simply $I_{p,q,r}$).

A homogeneous determinantal ideal $I \subset R$ is called a *standard determinantal* ideal if $r = \max(p, q)$. In other words, a homogeneous ideal $I \subset R$ of codimension c is called a *standard determinantal* ideal if $I = I_r(\mathcal{A})$ for some $r \times (r + c - 1)$ homogeneous matrix $\mathcal{A}$.

Remark 1.2.4. Let $\mathcal{A}$ be a t-homogeneous matrix of size $p \times q$. A classical result of J.A. Eagon and D.G. Northcott [21] asserts that, for any $r \leq t$, the ideal $I_r(\mathcal{A})$ generated by the $r \times r$ minors of $\mathcal{A}$ has

$$ht(I_r(\mathcal{A})) \leq (p - r + 1)(q - r + 1).$$

See [10, Theorem 2.1] for a proof. Therefore, for fixed p, q and r, determinantal ideals have maximal height among the ideals generated by the $r \times r$ minors of a t-homogeneous matrix of size $p \times q$ ($t \geq r$).

Definition 1.2.5. A homogeneous ideal $I \subset R$ is called a *symmetric determinantal* ideal if

(1) there exists a t-homogeneous symmetric matrix $\mathcal{A}$ of size $m \times m$ with entries in R, such that $I = I_t(\mathcal{A})$, and
(2) $ht(I) = \binom{m-t+2}{2}$.

Such ideal will be often denoted $I^s_{m,t}(\mathcal{A})$ (or simply $I^s_{m,t}$).

Remark 1.2.6. Let $\mathcal{A}$ be a t-homogeneous symmetric matrix of size $m \times m$. In [53, Theorem 2.1], T. Józefiak proved that, for any $t \leq m$, the ideal $I_t(\mathcal{A})$ generated by the $t \times t$ minors of $\mathcal{A}$ has

$$ht(I_r(\mathcal{A})) \leq \binom{m - t + 2}{2}.$$

Therefore, for a fixed t and m, symmetric determinantal ideals have maximal height among the ideals defined by $t \times t$ minors of a t-homogeneous symmetric matrix of size $m \times m$.

Definition 1.2.7. A subscheme $X \subset \mathbb{P}^{n+c}$ is said to be *standard determinantal* if its homogeneous saturated ideal $I(X)$ is a standard determinantal ideal. Therefore, a codimension c subscheme $X \subset \mathbb{P}^{n+c}$ is called a *standard determinantal* scheme if $I(X) = I_t(\mathcal{A})$ for some $t \times (t+c-1)$ homogeneous matrix $\mathcal{A}$. $X \subset \mathbb{P}^{n+c}$ is called a *good determinantal* scheme if, additionally, $\mathcal{A}$ contains a $(t-1) \times (t+c-1)$ submatrix (allowing a change of basis if necessary) whose ideal of maximal minors defines a scheme of codimension $c+1$.

A subscheme $X \subset \mathbb{P}^{n+c}$ is said to be *determinantal* if its homogeneous saturated ideal $I(X)$ is a determinantal ideal. Therefore, a codimension c subscheme $X \subset \mathbb{P}^{n+c}$ is called a *determinantal scheme* if there exist integers r, p, and q such that $c = (p-r+1)(q-r+1)$ and $I = I_r(\mathcal{A})$ for some r-homogeneous matrix $\mathcal{A}$ of size $p \times q$.

Example 1.2.8. (a) Complete intersections are examples of determinantal ideals.

(b) For any fixed integers $1 \leq p \leq q$ and for any r with $1 \leq r \leq p$, set $N = pq - 1$ and $R = K[x_{1,1}, x_{1,2}, \ldots, x_{1,q}, \ldots, x_{p,1}, x_{p,2}, \ldots, x_{p,q}]$. Set $X \subset \mathbb{P}^N = \mathrm{Proj}(R)$ be the determinantal scheme whose homogeneous saturated ideal is generated by the $r \times r$ minors of the matrix of indeterminates,

$$\mathcal{A} = \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,q} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,q} \\ \vdots & \vdots & \ldots & \vdots \\ x_{p,1} & x_{p,2} & \ldots & x_{p,q} \end{pmatrix}.$$

By [10, Theorem 2.5], X has

$$\mathrm{codim}(X) = \mathrm{depth}(I_r(\mathcal{A})) = (p-r+1)(q-r+1).$$

Therefore, X is ACM and determinantal.

(c) Let $X \subset \mathbb{P}^n$ be a rational normal curve defined as the image of the map

$$v_n : \mathbb{P}^1 \longrightarrow \mathbb{P}^n,$$
$$[a : b] \mapsto [a^n : a^{n-1}b : \cdots : ab^{n-1} : b^n].$$

It is a nice exercise to prove that the homogeneous ideal $I(X)$ of X is generated by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}.$$

Therefore, X is a good determinantal subscheme of $\mathbb{P}^n$.

Remark 1.2.9. Notice that a good determinantal scheme $X \subset \mathbb{P}^{n+c}$ is standard determinantal and the converse is true provided X is a generic complete intersection (cf. [61, Theorem 3.4] or [56, Proposition 3.2]).

One of the most important results on determinantal ideals is due to J.A. Eagon and M. Hochster. In [20], they proved the following theorem.

Theorem 1.2.10. *Determinantal schemes are arithmetically Cohen–Macaulay.*

Definition 1.2.11. A subscheme $X \subset \mathbb{P}^{n+c}$ is said to be *symmetric determinantal* if its homogeneous saturated ideal $I(X)$ is a symmetric determinantal ideal. Therefore, a codimension c subscheme $X \subset \mathbb{P}^{n+c}$ is called a *symmetric determinantal scheme* if there exist integers $t \leq m$ such that $c = \binom{m-t+2}{2}$ and $I(X) = I_t(\mathcal{A})$ for some t-homogeneous symmetric matrix $\mathcal{A}$ of size $m \times m$.

Example 1.2.12. (a) For any integer $1 \leq m$ and for any t with $1 \leq t \leq m$, set $N = \binom{m+1}{2} - 1$ and $R = K[\dots, x_{i,j}, \dots]_{1 \leq i \leq j \leq m}$. Set $X \subset \mathbb{P}^N = \operatorname{Proj}(R)$ be the symmetric determinantal scheme whose homogeneous saturated ideal $I(X)$ is generated by the $t \times t$ minors of the symmetric matrix of indeterminates of size $m \times m$,

$$\mathcal{A} = \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,m} \\ x_{1,2} & x_{2,2} & \dots & x_{2,m} \\ \vdots & \vdots & \dots & \vdots \\ x_{1,m} & x_{2,m} & \dots & x_{m,m} \end{pmatrix}.$$

By [53, Theorem 2.3], X has

$$\operatorname{codim}(X) = \operatorname{depth}(I_t(\mathcal{A})) = \binom{m-t+2}{2}.$$

Therefore, X is ACM and symmetric determinantal.

(b) Let $S \subset \mathbb{P}^5$ be a Veronese surface defined as the image of the map

$$v_{2,2} : \mathbb{P}^2 \longrightarrow \mathbb{P}^5,$$
$$[a : b : c] \mapsto [a^2 : ab : ac : b^2 : bc : c^2].$$

It is a nice exercise to prove that the homogeneous ideal $I(S)$ of S is generated by the 2×2 minors of the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Therefore, S is a symmetric determinantal subscheme of $\mathbb{P}^5$.

Remark 1.2.13. It is worth emphasizing that symmetric determinantal schemes do not occur for every codimension. There exist symmetric determinantal schemes of codimension c if and only if $c = \binom{b}{2}$ for some integer b. Moreover, complete intersections are a special case of symmetric determinantal schemes for the codimensions for which symmetric determinantal schemes do exist.

Cohen–Macaulayness of symmetric determinantal schemes was proved by R.E. Kutz. Indeed, in [62, Theorem 1], he proved the following.

Theorem 1.2.14. *Symmetric determinantal schemes are arithmetically Cohen–Macaulay.*

We now establish some further properties of determinantal schemes and symmetric determinantal schemes that we will need in this work.

Let $I \subset R$ be a homogeneous ideal. We let $\mu(I)$ denote the cardinality of a set of minimal generators of I. We have the following lemma.

Lemma 1.2.15. *Let $I^s_{m,t} \subset R$ be a symmetric determinantal ideal generated by the $t \times t$ minors of an $m \times m$ homogeneous symmetric matrix. Then, $\mu(I^s_{m,t}) = \binom{\binom{m}{t}+1}{2}$.*

Proof. Let X be a symmetric matrix of indeterminates of size $m \times m$. Then, $\mu(I^s_{m,t}) = \mu(I_t(X)) =$ number of distinct $t \times t$ minors of X. Therefore, we have

$$\begin{aligned} \mu(I^s_{m,t}) &= \mu(I_t(X)) \\ &= \frac{1}{2}\left[\binom{m}{t}\binom{m}{t} + \binom{m}{t}\right] \\ &= \binom{\binom{m}{t}+1}{2}. \end{aligned}$$ □

Now we are going to describe the generalized Koszul complexes associated with a codimension c standard determinantal scheme X. To this end, we denote by $\varphi : F \longrightarrow G$ the morphism of free graded R-modules of rank t and $t+c-1$, defined by the homogeneous matrix $\mathcal{A}$ associated with X. We denote by $\mathcal{C}_i(\varphi^*)$ the generalized Koszul complex,

$$\begin{aligned} \mathcal{C}_i(\varphi^*):\ 0 \longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \wedge^{i-1} G^* \otimes S_1(F^*) \\ \longrightarrow \cdots \longrightarrow \wedge^0 G^* \otimes S_i(F^*) \longrightarrow 0. \end{aligned}$$

Let $\mathcal{C}_i(\varphi^*)^*$ be the R-dual of $\mathcal{C}_i(\varphi^*)$. The map φ induces graded morphisms,

$$\mu_i : \wedge^{t+i} G^* \otimes \wedge^t F \longrightarrow \wedge^i G^*.$$

They can be used to splice the complexes $\mathcal{C}_{c-i-1}(\varphi^*)^* \otimes \wedge^{t+c-1} G^* \otimes \wedge^t F$ and $\mathcal{C}_i(\varphi^*)$ to a complex $\mathcal{D}_i(\varphi^*)$,

$$\begin{aligned} 0 &\longrightarrow \wedge^{t+c-1} G^* \otimes S_{c-i-1}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2} G^* \otimes S_{c-i-2}(F) \otimes \wedge^t F \\ &\longrightarrow \cdots \longrightarrow \wedge^{t+i} G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow \wedge^i G^* \otimes S_0(F^*) \longrightarrow \wedge^{i-1} G^* \otimes S_1(F^*) \\ &\longrightarrow \cdots \longrightarrow \wedge^0 G^* \otimes S_i(F^*) \longrightarrow 0. \end{aligned}$$

The complex $\mathcal{D}_0(\varphi^*)$ is called the *Eagon–Northcott complex* and the complex $\mathcal{D}_1(\varphi^*)$ is called the *Buchsbaum–Rim complex*. Let us rename the complex $\mathcal{C}_c(\varphi^*)$ as $\mathcal{D}_c(\varphi^*)$. Then we have the following well-known result.

Proposition 1.2.16. *Let $X \subset \mathbb{P}^{n+c}$ be a standard determinantal subscheme of codimension c associated with a graded minimal (i.e., $\operatorname{im}(\varphi) \subset \mathfrak{m}G$) morphism $\varphi : F \longrightarrow G$ of free R-modules of rank t and $t+c-1$, respectively. Set $M = \operatorname{coker}(\varphi^*)$. Then*

(1) *$\mathcal{D}_i(\varphi^*)$ is acyclic for $-1 \leq i \leq c$;*
(2) *$\mathcal{D}_0(\varphi^*)$ is a minimal free graded R-resolution of $R/I(X)$ and $\mathcal{D}_i(\varphi^*)$ is a minimal free graded R-resolution of length c of $S_i(M)$, $1 \leq i \leq c$;*
(3) *$K_X \cong S_{c-1}(M)$ up to degree shift. So, up to degree shift, $\mathcal{D}_{c-1}(\varphi^*)$ is a minimal free graded R-module resolution of K_X.*

Proof. See, for instance, [10, Theorem 2.16], and [22, Corollary A2.12 and Corollary A2.13]. □

Remark 1.2.17. By Proposition 1.2.16(2), any standard determinantal scheme $X \subset \mathbb{P}^{n+c}$ is ACM. Moreover, in codimension 2, the converse is true: If $X \subset \mathbb{P}^{n+2}$ is an ACM, closed subscheme of codimension 2, then it is standard determinantal. In fact, we have the following theorem.

Theorem 1.2.18 (Hilbert–Burch theorem). *Let $X \subset \mathbb{P}^{n+2}$ be an arithmetically Cohen–Macaulay subscheme of codimension 2. Then, there is an $r \times (r+1)$ homogeneous matrix $\mathcal{A}$, the Hilbert–Burch matrix whose maximal minors generate $I(X)$, giving rise to a minimal free R-resolution,*

$$0 \longrightarrow \oplus_{i=1}^{r} R(-b_i) \xrightarrow{\mathcal{A}} \oplus_{j=1}^{r+1} R(-a_j) \longrightarrow I(X) \longrightarrow 0.$$

Proof. See, for instance, [22, Theorem 20.15]. □

Remark 1.2.19. In codimension ≥ 3, it is no longer true that ACM schemes and standard determinantal schemes coincide. For instance, let $X = \{p_1, \dots, p_{10}\} \subset \mathbb{P}^3$ be a set of 10 general points. X is ACM because any 0-dimensional scheme is ACM but X is not standard determinantal. Indeed, the only way to define a 0-dimensional subscheme $X \subset \mathbb{P}^3$ of length 10 as the maximal minors of a $t \times (t+2)$ homogeneous matrix is by means of a 3×5 matrix with linear entries. Therefore, the dimension of the family of 0-dimensional schemes $X \subset \mathbb{P}^3$ of length 10 defined by the maximal minors of a 3×5 matrix with linear entries is 27, while the dimension of the family of sets of 10 general points in $\mathbb{P}^3$ is 30.

The homogeneous matrix $\mathcal{A}$ associated with a standard determinantal scheme $X \subset \mathbb{P}^{n+c}$ of codimension c also defines an injective morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of locally free $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules of rank t and $t+c-1$. Since the construction of the generalized Koszul complexes globalizes, we can also associate with φ^* the *Eagon–Northcott complex* of $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules,

$$0 \longrightarrow \wedge^{t+c-1}\mathcal{G}^* \otimes S_{c-1}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \wedge^{t+c-2}\mathcal{G}^* \otimes S_{c-2}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \cdots$$
$$\longrightarrow \wedge^{t}\mathcal{G}^* \otimes S_{0}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+c}} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

and the *Buchsbaum–Rim complex* of locally free $\mathcal{O}_{\mathbb{P}^{n+c}}$-modules,

$$0 \longrightarrow \wedge^{t+c-1}\mathcal{G}^* \otimes S_{c-2}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \wedge^{t+c-2}\mathcal{G}^* \otimes S_{c-3}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \cdots$$
$$\longrightarrow \wedge^{t+1}\mathcal{G}^* \otimes S_{0}(\mathcal{F}) \otimes \wedge^t \mathcal{F} \longrightarrow \mathcal{G}^* \xrightarrow{\varphi^*} \mathcal{F}^* \longrightarrow \tilde{M} \longrightarrow 0.$$

Since the degeneracy locus of φ^* has codimension c (In fact, $I(\varphi^*) = I(\mathcal{A}) = I(X)$), these two complexes are acyclic. Moreover, the kernel of φ^* is called the *first Buchsbaum–Rim sheaf* associated with φ^*.

Let $X \subset \mathbb{P}^{n+c}$ be a standard (resp., good) determinantal scheme of codimension $c \geq 2$ defined by the vanishing of the maximal minors of a $t \times (t + c - 1)$ homogeneous matrix $\mathcal{A}$. The matrix $\mathcal{A}$ defines a degree 0 morphism

$$\varphi : F = \oplus_{i=1}^{t} R(b_i) \longrightarrow G = \oplus_{j=0}^{t+c-2} R(a_j)$$

of free graded R-modules of rank t and $t + c - 1$, respectively. Hence,

$$\mathcal{A} = (f_{ji})_{i=1,\dots,t}^{j=0,\dots,t+c-2},$$

where $f_{ji} \in K[x_0, \dots, x_{n+c}]$ are homogeneous polynomials of degree $a_j - b_i$ with $b_1 \leq \cdots \leq b_t$ and $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. We assume, without loss of generality, that $\mathcal{A}$ is minimal; i.e., $f_{ji} = 0$ for all i, j with $b_i = a_j$. If we let $u_{j,i} = a_j - b_i$ for all $j = 0, \dots, t + c - 2$ and $i = 1, \dots, t$, the matrix

$$\mathcal{U} = (u_{j,i})_{i=1,\dots,t}^{j=0,\dots,t+c-2}$$

is called the *degree matrix* associated with X. We have the following lemma.

Lemma 1.2.20. *The matrix $\mathcal{U}$ has the following properties:*

(1) *for every j and i, $u_{j,i} \leq u_{j+1,i}$ and $u_{j,i} \geq u_{j,i+1}$;*
(2) *for every $i = 1, \dots, t$, $u_{i-1,i} = a_{i-1} - b_i > 0$.*

And vice versa, given a degree matrix $\mathcal{U}$ of integers verifying (1) *and* (2)*, there exists a codimension c standard* (*resp., good*) *determinantal scheme $X \subset \mathbb{P}^{n+c}$ with associated degree matrix $\mathcal{U}$.*

Proof. The first condition is obvious. For the second one, we only need to observe that if for some $i = 1, \dots, t$, we have $u_{i-1,i} \leq 0$, then in the matrix $\mathcal{A}$ we have $f_{j,k} = 0$ for $j \leq i - 1$ and $k \geq i$. But this would imply that the minor which is obtained by deleting the last $c - 1$ columns has to be zero, contradicting the minimality of $\mathcal{A}$.

The converse is trivial. Indeed, given a matrix of integers, $\mathcal{U}$, satisfying (1) and (2), we can consider the standard (resp., good) determinantal scheme $X \subset \mathbb{P}^{n+c}$ of codimension c associated with the homogeneous matrix

$\mathcal{A} =$

$$\begin{pmatrix} x_0^{u_{t+c-2,t}} & x_1^{u_{t+c-3,t}} & \cdots & \cdots & x_{c-1}^{u_{t-1,t}} & 0 & 0 & \cdots & \cdots \\ 0 & x_0^{u_{t+c-3,t-1}} & x_1^{u_{t+c-4,t-1}} & \cdots & \cdots & x_{c-1}^{u_{t-2,t-1}} & 0 & 0 & \cdots \\ 0 & 0 & x_0^{u_{t+c-4,t-2}} & x_1^{u_{t+c-5,t-2}} & \cdots & \cdots & x_{c-1}^{u_{t-3,t-2}} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

respectively, $\mathcal{A} =$

$$\begin{pmatrix} x_0^{u_{t+c-2,t}} & x_1^{u_{t+c-3,t}} & \cdots & \cdots & x_{c-1}^{u_{t-1,t}} & 0 & 0 & \cdots & \cdots \\ x_c^{u_{t+c-2,t-1}} & x_0^{u_{t+c-3,t-1}} & x_1^{u_{t+c-4,t-1}} & \cdots & \cdots & x_{c-1}^{u_{t-2,t-1}} & 0 & 0 & \cdots \\ 0 & x_c^{u_{t+c-3,t-2}} & x_0^{u_{2,t-2}} & x_1^{u_{t+c-5,t-2}} & \cdots & \cdots & x_{c-1}^{u_{t-3,t-2}} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Up to reordering, we easily check that the degree matrix associated with X is $\mathcal{U}$. $\square$

Given integers $b_1, \ldots, b_t$ and $a_0, a_1, \ldots, a_{t+c-2}$, we denote by $W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ (resp., $W_s(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$) the locus of good (resp., standard) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension $c \geq 2$ defined by the maximal minors of a homogeneous matrix $\mathcal{A} = (f_{ji})_{i=1,\ldots,t}^{j=0,\ldots,t+c-2}$ where $f_{ji} \in K[x_0, \ldots, x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. Clearly, $W(\underline{b}; \underline{a}) \subset W_s(\underline{b}; \underline{a})$. Moreover we have the following corollary.

Corollary 1.2.21. *Assume $b_1 \leq \cdots \leq b_t$ and $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. We have that $W(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $W_s(\underline{b}; \underline{a}) \neq \emptyset$ if and only if $u_{i-1,i} = a_{i-1} - b_i > 0$ for $i = 1, \ldots, t$.*

Proof. It easily follows from Lemma 1.2.20. $\square$

Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of codimension $c \geq 2$ defined by the homogeneous matrix $\mathcal{A} = (f_{ji})_{i=1,\ldots t}^{j=0,\ldots,t+c-2}$. It is well known that by successively deleting columns from the right-hand side of $\mathcal{A}$, and taking maximal minors, one gets a flag of determinantal subschemes,

$$(\mathbf{X}.) : X = X_c \subset X_{c-1} \subset \cdots \subset X_2 \subset \mathbb{P}^{n+c}, \tag{1.5}$$

where each $X_{i+1} \subset X_i$ (with ideal sheaf $\mathcal{I}_{X_{i+1}|X_i} := \mathcal{I}_i$) is of codimension 1, $X_i \subset \mathbb{P}^{n+c}$ is of codimension i ($i = 2, \ldots, c$) and where there exist $\mathcal{O}_{X_i}$-modules $\mathcal{M}_i$ fitting into short exact sequences,

$$0 \to \mathcal{O}_{X_i}(-a_{t+i-1}) \to \mathcal{M}_i \to \mathcal{M}_{i+1} \to 0 \quad \text{for} \;\; 2 \leq i \leq c-1, \tag{1.6}$$

such that $\mathcal{I}_i(a_{t+i-1})$ is the $\mathcal{O}_{X_i}$-dual of $\mathcal{M}_i$ for $2 \leq i \leq c$, and $\mathcal{M}_2$ is a twist of the canonical module of X_2; cf. (4.5)–(4.8) for details.

Remark 1.2.22. Assume that $b_1 \leq \cdots \leq b_t$ and $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. If X is general in $W(\underline{b};\underline{a})$ and $u_{i-\min(\alpha,t),i} = a_{i-\min(\alpha,t)} - b_i \geq 0$ for $\min(\alpha,t) \leq i \leq t$, then $X_j = \mathrm{Proj}(D_j)$, for all $j = 2,\dots,c$, is non-singular except for a subset of codimension at least $\min\{2\alpha-1, j+2\}$; i.e.,

$$\mathrm{codim}_{X_j} \mathrm{Sing}(X_j) \geq \min\{2\alpha-1, j+2\}. \tag{1.7}$$

This follows from the theorem in [12], arguing as in [12, Example 2.1]. In particular, if $\alpha \geq 3$, we get that for each $i > 0$, the closed embeddings $X_i \subset \mathbb{P}^{n+c}$ and $X_{i+1} \subset X_i$ are local complete intersections outside some set Z_i of codimension at least $\min(4, i+1)$ in X_{i+1} ($\mathrm{depth}_{Z_i} \mathcal{O}_{X_{i+1}} \geq \min(4, i+1)$).

Moreover, taking $\alpha = 1$, we deduce from (1.7) that a general X in $W(\underline{b};\underline{a})$ is reduced provided $a_{i-1} > b_i$ for all $1 \leq i \leq t$. This means (see Corollary 1.2.21) that a nonempty $W(\underline{b};\underline{a})$ always contains a reduced determinantal scheme. This remark improves [11, Proposition 2.7].

1.3 CI-liaison and G-liaison

In this section, we recall the definitions and basic facts on CI-liaison and G-liaison needed in the sequel. Liaison theory has been a very active area of research during the last decades. Historically liaison began in the nineteenth century as a tool to study curves in projective spaces and it goes back at least to the work of M. Noether [73], F. Severi [82, 83], and F.S. Macaulay [63]. The initial idea was to start with a curve in $\mathbb{P}^3$ and to study its residual in a complete intersection. It turns out that a lot of information can be carried over from a curve to its residual and vice versa. It was hoped that using this process of linking curves, one could always pass to a "simpler" curve, namely a complete intersection (in some sense, the simplest curve), and so it would be possible to get information about the original curve from this complete intersection.

In the first half of the twentieth century, P. Dubreil [19], R. Apéry [3], and F. Gaeta [28], among others, contributed to the initial development of liaison theory. But it was not until the work of C. Peskine and L. Szpiro [75] in 1974 that liaison was established as a modern discipline. Roughly speaking, liaison is an equivalence relation among subschemes of a given dimension d in a projective space $\mathbb{P}^n$ (or ideals in $K[x_0, x_1, \dots, x_n]$) and it involves the study of the properties that are shared by two schemes whose union is well understood. Two subschemes of the same dimension are said to be *directly CI-linked* if their union is a complete intersection and are said to be *directly G-linked* if their union is an AG scheme (see Definition 1.3.1 for the precise definition). *CI-liaison* (resp., *G-liaison*) is the equivalence relation generated by CI-links (resp., G-links). Notice that as complete intersections are particular cases of AG schemes, any two subschemes which are in the same CI-liaison class are also in the same G-liaison class.

Most of the work on liaison theory has been done for subschemes of codimension 2 in projective spaces where complete intersection and AG schemes coincide.

When we pass to arbitrary codimension, one has to decide whether to view the codimension 2 case as a part of CI-liaison theory or G-liaison theory. Different authors have tried to generalize the results of CI-liaison for codimension 2 schemes to schemes of higher codimension. For instance, C. Huneke and B. Ulrich [52] (see also [56]) proved that some interesting results in codimension 2 do not hold when we link higher-codimensional ideals by complete intersections. In [79], P. Schenzel realized that many of the basic results were also valid in arbitrary codimension when one links using AG schemes instead of complete intersections (see also the work of E.S. Golod [30]) and the work [56] strongly suggests that the idea of linking using AG schemes is indeed a natural generalization to higher codimension of the idea of linking using complete intersections.

Definition 1.3.1 (See also [56, Definitions 2.3, 2.4, and 2.10]). We say that two subschemes V_1 and V_2 of $\mathbb{P}^n$ are (*algebraically*) *directly CI-linked* or simply (*algebraically*) *CI-linked* (resp., (*algebraically*) *directly G-linked* or simply (*algebraically*) *G-linked*), by a complete intersection (resp., an AG) subscheme $X \subset \mathbb{P}^n$ if $I(X) \subset I(V_1) \cap I(V_2)$ and we have $I(X) : I(V_1) = I(V_2)$ and $I(X) : I(V_2) = I(V_1)$.

We will say that V_1 and V_2 are *directly linked* when it does not matter if the linking scheme is an AG or a complete intersection scheme.

Remark 1.3.2. The definition of direct links that we have given here does not coincide with the original definition given by C. Peskine and L. Szpiro [75], A.P. Rao [76], and P. Schenzel [79]. They said that $V_1, V_2 \subset \mathbb{P}^n$ are *algebraically linked* by X if and only if

(1) $I(X) \subset I(V_1) \cap I(V_2)$,
(2) V_1 and V_2 are equidimensional schemes without embedded components,
(3) $\mathcal{I}_{V_1}/\mathcal{I}_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_2}, \mathcal{O}_X)$ (or, equivalently, $\mathrm{Hom}_R(R/I(V_2), R/I(X)) \cong I(V_1)/I(X)$), and
(4) $\mathcal{I}_{V_2}/\mathcal{I}_X \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{V_1}, \mathcal{O}_X)$ (or, equivalently, $\mathrm{Hom}_R(R/I(V_1), R/I(X)) \cong I(V_2)/I(X)$).

The equivalence between both definitions was first observed by P. Schenzel in [79]. To see the equivalence, the main point is that there are natural isomorphisms,

$$[I(V_1) : I(X)]/I(X) \cong \mathrm{Hom}_R(R/I(V_1), R/I(X))$$

and

$$[I(V_2) : I(X)]/I(X) \cong \mathrm{Hom}_R(R/I(V_2), R/I(X)).$$

Another way of saying this is that $I(V_2)$ is the annihilator of $I(V_1)$ in $R/I(X)$ and $I(V_1)$ is the annihilator of $I(V_2)$ in $R/I(X)$.

If V_1 and V_2 do not share any common component, the definition of direct linkage has a clear geometric meaning. In fact, it is equivalent to $X = V_1 \cup V_2$ as schemes (or, equivalently, $I(X) = I(V_1) \cap I(V_2)$) and in this case we say that V_1 and V_2 are directly *geometrically CI-linked* (resp., *geometrically G-linked*).

Example 1.3.3. (a) A simple example of directly geometrically CI-linked schemes is the following one: Let C_1 be a twisted cubic in $\mathbb{P}^3$ and let C_2 be a secant line to C_1. The union of C_1 and C_2 is a degree 4 curve which is the complete intersection X of two quadrics Q_1 and Q_2. So C_1 and C_2 are directly CI-linked by the complete intersection X. More precisely, we take $C_1 \subset \mathbb{P}^3$, the twisted cubic with the homogeneous ideal

$$I(C_1) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \subset K[x_0, x_1, x_2, x_3],$$

and the complete intersection ideal

$$I(X) = (x_1^2 - x_2x_0 + x_2^2 - x_1x_3, x_1^2 - x_0x_2 + 2x_2^2 - 2x_1x_3) \subset I(C_1),$$

the residual to C_1 in X is the line C_2 defined by $I(C_2) = (x_1, x_2)$.

(b) As a simple example of directly geometrically G-linked schemes we have the following: We consider a set $Y_1 \subset \mathbb{P}^3$ of four points in linear general position and a sufficiently general point Y_2. Since $X = Y_1 \cup Y_2$ is an AG subscheme, Y_1 and Y_2 are directly G-linked.

(c) Let $C \subset \mathbb{P}^4$ be the rational normal quartic with homogeneous ideal

$$I(C) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_0x_4 - x_3x_1, x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2).$$

We consider the elliptic curve $X \subset \mathbb{P}^4$ with homogeneous ideal

$$I(X) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_0x_4 - x_3x_1, x_1x_3 - x_2^2, x_1x_4 - x_2x_3).$$

X is an AG curve and the residual to C in X is the line $L \subset \mathbb{P}^4$ with homogeneous ideal $I(L) = (x_0, x_1, x_2)$. In fact, $I(X) : I(C) = (x_0, x_1, x_2)$. So, C and L are geometrically directly G-linked.

(d) Let $I(X) = (x_0x_1, x_0 + x_1) = (x_0^2, x_0 + x_1) = (x_1^2, x_0 + x_1) \subset K[x_0, x_1, x_2, x_3]$ and let $I(C) = (x_0, x_1)$. Then, we have $[I(X) : I(C)] = I(C)$. That is, the line $C \subset \mathbb{P}^3$ is self-CI-linked by $I(X)$.

The question of whether a subscheme $X \subset \mathbb{P}^n$ can be self-CI-linked (resp., self-G-linked) is an interesting and difficult one that has been addressed by several authors, see, e.g., [7], [16], [27], [55], [65], and [76].

Definition 1.3.4. Let V_1 and $V_2 \subset \mathbb{P}^n$ be two equidimensional schemes without embedded components. We say that V_1 and V_2 are in the same *CI-liaison class* (resp., *G-liaison class*) if and only if there exists a sequence of schemes $Y_1, \ldots, Y_r$ such that Y_i is *directly CI-linked* (resp., *directly G-linked*) to Y_{i+1} and such that $Y_1 = V_1$ and $Y_r = V_2$. If V_1 is linked to V_2 in two steps by complete intersection (resp., AG) schemes, we say that they are *CI-bilinked* (resp., *G-bilinked*).

In other words *CI-liaison*(resp., *G-liaison*) is the equivalence relation generated by direct CI-linkage (resp., direct G-linkage) and, roughly speaking, liaison theory is the study of these equivalence relations and the corresponding equivalence classes.

In codimension 2 CI-liaison and G-liaison generate the same equivalence relation, since complete intersection subschemes and AG subschemes coincide. In higher codimension it is no longer true. Indeed, a simple counterexample is the following: Consider a set X of four points in $\mathbb{P}^3$ in linear general position. By Example 1.3.3(b) we can G-link X to a single point. Therefore, X is glicci (see Definition 1.3.5). On the other hand, it follows from [52, Corollary 5.13] that X is not licci (see Definition 1.3.5).

As remarked in the introduction, we will see in Chapter 2 that for codimension 2 subschemes $X \subset \mathbb{P}^n$, being in the CI-liaison class of a complete intersection is equivalent to being ACM. In higher codimension, this is no longer true.

Definition 1.3.5. A subscheme $X \subset \mathbb{P}^n$ is said to be *licci* if it is in the CI-liaison class of a complete intersection.

Analogously, we say that a subscheme $X \subset \mathbb{P}^n$ is *glicci* if it is in the G-liaison class of a complete intersection.

Example 1.3.6. (a) Let $C \subset \mathbb{P}^3$ be the twisted cubic parameterized by $(s^3,\ s^2t,\ st^2,\ t^3)$. We easily compute its homogeneous ideal and we get

$$I(C) = (x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2) \subset K[x_0, x_1, x_2, x_3].$$

Therefore, C is contained in the complete intersection of two quadrics $X \subset \mathbb{P}^3$ defined by $I(X) = (x_0x_3 - x_1x_2, x_0x_2 - x_1^2) \subset I(C)$. We verify that $I(X) : I(C) = (x_0, x_1)$ defining a line $L \subset \mathbb{P}^3$. Thus C and L are CI-linked by X and C is licci.

(b) Let $C \subset \mathbb{P}^4$ be the rational normal quartic with

$$I(C) = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2) \subset K[x_0, x_1, x_2, x_3, x_4].$$

Let $X \subset \mathbb{P}^4$ be the elliptic quintic with

$$I(X) = (x_2^2, x_0^2, x_1x_2, x_0x_2 - x_1^2, x_0x_1) \subset K[x_0, x_1, x_2, x_3, x_4].$$

X is an AG subscheme and its homogeneous ideal $I(X)$ is generated by the Pfaffians of the skew symmetric matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & x_0 & x_1 \\ 0 & 0 & x_0 & x_1 & x_2 \\ 0 & -x_0 & x_1 & x_2 & 0 \\ -x_0 & -x_1 & -x_2 & 0 & 0 \\ -x_1 & -x_2 & 0 & 0 & 0 \end{pmatrix}.$$

Since $I(X) \subset I(C)$ and $I(X) : I(C) = (x_0, x_1, x_2)$, we can use X to G-link C and the line L with homogeneous ideal $I(L) = (x_0, x_1, x_2)$, and to conclude that the rational normal quartic C is glicci.

In order for meaningful applications of G-liaison theory to be found, we need useful constructions of Gorenstein ideals. We will end this section describing some constructions which have already been proved useful in the context of G-liaison theory.

The first way to construct AG schemes is to start with two ACM schemes whose union is AG and which have no common component, and look at the intersection of those schemes. We call this process the *sum of geometrically linked ideals*. More precisely, we have (see [75]) the following theorem.

Theorem 1.3.7. *Let X and Y be ACM subschemes of codimension c which have no common component, and assume that $I(X) \cap I(Y)$ is AG. Then $I(X) + I(Y)$ is AG of codimension $c+1$.*

Proof. It is enough to apply the mapping cone process to the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R(-t) & \longrightarrow & F_c \oplus G_c & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_{c-1} & \longrightarrow & F_{c-1} \oplus G_{c-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_2 & \longrightarrow & F_2 \oplus G_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_1 & \longrightarrow & F_1 \oplus G_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I(X) \cap I(Y) & \longrightarrow & I(X) \oplus I(Y) & \longrightarrow & I(X) + I(Y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the H_i give a minimal free R-resolution of the AG ideal $I(X) \cap I(Y)$, the F_i give a minimal free R-resolution of $I(X)$, and the G_i give a minimal free R-resolution of $I(Y)$. □

Example 1.3.8. Let $X \subset \mathbb{P}^n$ be an ACM codimension 2 subscheme with h-vector

$$1\ 2\ 3\ 4\ \ldots\ s-1\ 0\ 0\ \ldots\ .$$

We take F and G as two general forms of degree s and we denote by $Y \subset \mathbb{P}^n$ the codimension 2 ACM subscheme CI-linked to X via the complete intersection (F, G). Then $X \cap Y \subset \mathbb{P}^n$ is a codimension 3 AG subscheme with h-vector

$$1\ 3\ 6\ \ldots\ \binom{s}{2}\ \binom{s+1}{2}\ \binom{s}{2}\ \ldots\ 6\ 3\ 1\ 0\ 0\ \ldots .$$

Definition 1.3.9. A subscheme $X \subset \mathbb{P}^n$ satisfies the condition G_r if every localization of $R/I(X)$ of dimension $\leq r$ is a Gorenstein ring. G_r is sometimes referred to as "Gorenstein in codimension $\leq r$"; i.e., the nonlocally Gorenstein locus has codimension $\geq r+1$. In particular, G_0 is generically Gorenstein.

Definition 1.3.10. Let $S \subset \mathbb{P}^n$ be an ACM subscheme and let $F \in K[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial of degree d not vanishing on any irreducible component of S (i.e., $I(S) : F = I(S)$). Then H_F is the divisor on S cut out by F and we call it the *hypersurface section* of S cut out by F. As a subscheme of $\mathbb{P}^n$, H_F is defined by the ideal $I(S) + (F)$. Note that this ideal is saturated, since S is ACM.

Theorem 1.3.11. *Let $S \subset \mathbb{P}^n$ be an ACM subscheme satisfying G_1 and let K be a canonical divisor on S and H the hyperplane section. Then, any effective divisor X of the linear system $|dH - K|$, viewed as a subscheme of $\mathbb{P}^n$, is AG.*

Proof. Choose a sufficiently large integer ℓ such that there is a regular section of $\omega_S(\ell)$ defining a twisted canonical divisor Y.

Let $F \in I(Y) \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d + \ell$ such that F does not vanish on any irreducible component of S, and let H_F be the corresponding hypersurface section. Then X is linearly equivalent to the effective divisor $H_F - Y$, and we have the isomorphisms

$$\begin{aligned} \mathcal{I}_{X|S}(d) &\cong \mathcal{O}_S((d+\ell)H - X) \\ &\cong \mathcal{O}_S(Y) \\ &\cong \omega_S(\ell). \end{aligned}$$

Suppose that $S \subset \mathbb{P}^n$ has codimension r. Since S is ACM, the dual of the minimal free R-resolution of $I(S)$ is a minimal free resolution for $H^0_*(\omega_S)$, and the last free module in this resolution has rank 1. We thus have the following exact diagram (up to twist – what are important are the ranks):

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & 0 & & & R & & \\
 & \downarrow & & & \downarrow & & \\
 & F_r & & & F_1^* & & \\
 & \downarrow & & & \downarrow & & \\
 & F_{r-1} & & & F_2^* & & \\
 & \downarrow & & & \downarrow & & \\
 & \vdots & & & \vdots & & \\
 & \downarrow & & & \downarrow & & \\
 & F_1 & & & F_r^* & & \\
 & \downarrow & & & \downarrow & & \\
0 \to & I(S) & \to & I(X) & \to \; H^0_*(\omega_S)(\ell - d) \cong K_S & \to & 0. \\
 & \downarrow & & & \downarrow & & \\
 & 0 & & & 0 & &
\end{array}$$

The Horseshoe lemma then shows that $I(X)$ has a free R-resolution in which the last free module has rank 1. Since $\operatorname{codim} X = r+1$, this last free module cannot split off, so X is AG. Note that this shows that any element in the linear system of X is AG. □

Theorem 1.3.12. *Let $S \subset \mathbb{P}^n$ be an ACM subscheme satisfying G_1 and let $C \subset S$ be an effective divisor. Take any effective divisor C_1 in the linear system $|C+tH|$ where H is a hyperplane section of S and $t \in \mathbb{Z}$. Then, C and C_1 are G-bilinked. (Notice that if $t=0$, then C and C_1 are linearly equivalent.)*

Proof. Let us assume that S is smooth. For the general case we refer the reader to [56, Corollary 5.12].

Let Y be an effective twisted canonical divisor. Choose an integer $0 \ll a \in \mathbb{Z}$ such that the homogenous ideal $I(Y)$ of Y contains a form A of degree a not vanishing on any irreducible component of S. Hence, $H_A - Y$ is an effective divisor on S. Since $C_1 \in |C+tH|$, there exist homogeneous forms F and G with $\deg(F) = \deg(G)+t$ and a divisor D such that $H_F = C_1 + D$ and $H_G = C + D$. In particular, $F \in I(C_1)$ and $G \in I(C)$. By Theorem 1.3.11, the effective divisors $H_{AF} - Y$ and $H_{AG} - Y$ are AG and one can easily check that

$$\begin{aligned}(H_{AF} - Y) - C_1 &= (H_A - Y) + (H_F - C_2)\\ &= (H_A - Y) + D\end{aligned}$$

and

$$\begin{aligned}(H_{AG} - Y) - C &= (H_A - Y) + (H_G - C)\\ &= (H_A - Y) + D.\end{aligned}$$

Therefore, C_1 is directly linked to $(H_A - Y) + D$ by the Gorenstein subscheme $H_{AF} - Y$ and C is directly linked to $(H_A - Y) + D$ by the Gorenstein subscheme $H_{AG} - Y$. So, we conclude that C and C_1 are G-linked in two steps. □

Theorem 1.3.12 shows that G-liaison is a theory about divisors on ACM schemes, just as Hartshorne [42] has shown that CI-liaison is a theory about divisors on complete intersections. It is fair to say that most of the results about G-liaison discovered in the last few years use this result either directly or at least indirectly (see, for instance, [13], [14], [15]). Let us end this section illustrating with a nice example how to use Theorem 1.3.12.

Theorem 1.3.12 motivates the following definition.

Definition 1.3.13. Let $X \subset \mathbb{P}^n$ be a smooth scheme. We say that an *effective divisor* C on X is *minimal* if there is no effective divisor in the linear system $|C - H|$, where H is a hyperplane section divisor of X.

Terminology 1.3.14. To say that a statement holds for a general point of a projective variety Y means that there exists a countable union Z of proper subvarieties of Y such that the statement holds for every $x \in Y \setminus Z$. In particular, we say that a statement holds for a *general* surface $X \subset \mathbb{P}^4$ with Hilbert polynomial $p(t)$ if the statement holds for a general point of an irreducible component of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^4)$.

From now on, unless otherwise specified the word general, when refers to elements of projective varieties, will have this meaning. We have the following theorem.

Theorem 1.3.15. *All ACM curves $C \subset \mathbb{P}^4$ lying on a general, smooth, rational, ACM surface $S \subset \mathbb{P}^4$ are glicci; i.e., they belong to the G-liaison class of a complete intersection.*

Sketch of the proof. According to the classification of general, smooth, rational, ACM surfaces, S is either

(1) a cubic scroll: $S = Bl_{\{p_1\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|2E_0 - E_1|$, $\deg(S) = 3$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^2 = \langle E_0; E_1\rangle$, or

(2) a Del Pezzo surface: $S = Bl_{\{p_1,\dots,p_5\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|3E_0 - \sum_{i=1}^5 E_i|$, $\deg(S) = 4$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^6 = \langle E_0; E_1, \dots, E_5\rangle$, or

(3) a Castelnuovo surface: $S = Bl_{\{p_1,\dots,p_8\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - 2E_1 - \sum_{i=2}^8 E_i|$, $\deg(S) = 5$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^9 = \langle E_0; E_1, \dots, E_8\rangle$, or

(4) a Bordiga surface: $S = Bl_{\{p_1,\dots,p_{10}\}}(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$ by means of the linear system $|4E_0 - \sum_{i=1}^{10} E_i|$, $\deg(S) = 6$, and $\mathrm{Pic}(S) \cong \mathbb{Z}^{11} = \langle E_0; E_1, \dots, E_{10}\rangle$.

For each general, smooth, rational, ACM surface, we classify the minimal ACM curves C on S (see [56, Section 8]). Finally, we check that each minimal ACM curve C on S is glicci by direct examination and we apply Theorem 1.3.12 to conclude that any other ACM curve C on S is glicci. □

Chapter 2

CI-liaison and G-liaison of Standard Determinantal Ideals

It is a classical result, originally proved by F. Gaeta [28], in 1948, and re-proved in modern language by C. Peskine and L. Szpiro [75], that every ACM codimension 2 subscheme $X \subset \mathbb{P}^n$ can be CI-linked in a finite number of steps to a complete intersection subscheme or, equivalently, all codimension 2, ACM subschemes $X \subset \mathbb{P}^n$ are licci. In the first section of this chapter we will sketch a proof of Gaeta's theorem which can be viewed as a first result on standard determinantal subschemes, because if $X \subset \mathbb{P}^n$ is standard determinantal, then X is ACM and, moreover, the Hilbert–Burch theorem states that, in codimension 2, the converse is also true.

The goal of Section 2.2 of this chapter is to see that in the CI-liaison context Gaeta's theorem does not generalize well to subschemes $X \subset \mathbb{P}^n$ of higher codimension. We will introduce some graded modules which are CI-liaison invariants, and using them, we will prove the existence of infinitely many different CI-liaison classes containing standard determinantal schemes. More precisely, we will see that an ACM curve $C_t \subset \mathbb{P}^4$, defined by the maximal minors of a $t \times (t+1)$ matrix with linear entries, has a linear resolution,

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0$$

and $C_t, C_{t'}$ belong to different CI-liaison classes provided $t \neq t'$ (Corollary 2.2.13).

In the last section of this chapter we generalize Gaeta's theorem and we prove that standard determinantal schemes are glicci (Theorem 2.3.1). Since in codimension 2, ACM schemes are standard determinantal, and AG schemes and complete intersection schemes coincide, this result is indeed a full generalization of Gaeta's theorem.

2.1 CI-liaison class of Cohen–Macaulay codimension 2 ideals

In 1940s, R. Apéry [3] announced, and F. Gaeta [28] proved, that the initial idea of M. Noether of studying curves $C \subset \mathbb{P}^3$ by looking at their residual in a complete intersection works only for ACM curves in $\mathbb{P}^3$; i.e., a curve $C \subset \mathbb{P}^3$ is in the CI-liaison class of a complete intersection if and only if C is ACM. Later, in 1974, C. Peskine and L. Szpiro [75] set the modern base of liaison theory and they proved that ACM codimension 2 subschemes of $\mathbb{P}^n$ form a CI-liaison class. The goal of this section is to sketch a proof of this result. To this end, we will begin investigating the relation between CI-linked and G-linked subschemes. In particular, we will compare the free R-resolution of directly CI-linked and G-linked ideals and the deficiency modules of CI-linked and G-linked subschemes.

Proposition 2.1.1. *Let $X, Y \subset \mathbb{P}^n$ be two subschemes of codimension c directly G-linked by an AG subscheme $Z \subset \mathbb{P}^n$ with sheafified minimal free resolution,*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t) \longrightarrow \mathcal{F}_{c-1} \longrightarrow \cdots \\ \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{I}_Z \longrightarrow 0. \end{aligned}$$

Assume that X is locally Cohen–Macaulay and let

$$\begin{aligned} 0 \longrightarrow \mathcal{G}_c \longrightarrow \mathcal{G}_{c-1} \longrightarrow \cdots \\ \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{I}_X \longrightarrow 0 \end{aligned}$$

be a locally free resolution of $\mathcal{I}_X$. Then there is a locally free resolution of $\mathcal{I}_Y$ of the following type:

$$\begin{aligned} 0 \longrightarrow \mathcal{G}_1^\vee(-t) \longrightarrow \mathcal{F}_1^\vee(-t) \oplus \mathcal{G}_2^\vee(-t) \longrightarrow \cdots \\ \longrightarrow \mathcal{F}_{c-1}^\vee(-t) \oplus \mathcal{G}_c^\vee(-t) \longrightarrow \mathcal{I}_Y \longrightarrow 0. \end{aligned}$$

Proof. Since X and Y are directly G-linked by Z, we have

$$\begin{aligned} \mathcal{I}_X/\mathcal{I}_Z &\cong \mathcal{H}om(\mathcal{O}_Y, \mathcal{O}_Z) \\ &\cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{I}_Z) \\ &\cong \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-t)) \\ &\cong \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-n-1))(n+1-t) \\ &\cong \omega_Y(n+1-t). \end{aligned}$$

So, we have an exact sequence,

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_X \longrightarrow \omega_Y(n+1-t) \longrightarrow 0. \tag{2.1}$$

The sheafified minimal free resolution for $\mathcal{I}_Z$ and the locally free resolution of $\mathcal{I}_X$ combine with the exact sequence (2.1) to form a commutative diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^n}(-t) & \longrightarrow & \mathcal{G}_c & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F}_{c-1} & \longrightarrow & \mathcal{G}_{c-1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F}_1 & \longrightarrow & \mathcal{G}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{I}_X & \longrightarrow & \omega_Y(n+1-t) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the mapping cone process applied to this last diagram gives us a locally free resolution of $\omega_Y(n+1-t)$,

$$\begin{aligned}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t) \longrightarrow \mathcal{G}_c \oplus \mathcal{F}_{c-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \oplus \mathcal{G}_2 \\
\longrightarrow \mathcal{G}_1 \longrightarrow \omega_Y(n+1-t) \longrightarrow 0.
\end{aligned}$$

Dualizing and twisting by $\cdot \otimes \mathcal{O}_{\mathbb{P}^n}(-t)$, we obtain the resolution of $\mathcal{O}_Y$,

$$\begin{aligned}
0 \longrightarrow \mathcal{G}_1^\vee(-t) \longrightarrow \mathcal{F}_1^\vee(-t) \oplus \mathcal{G}_2^\vee(-t) \longrightarrow \cdots \longrightarrow \mathcal{F}_{c-1}^\vee(-t) \oplus \mathcal{G}_c^\vee(-t) \\
\longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{E}xt^c_{\mathcal{O}_{\mathbb{P}^n}}(\omega_Y, \omega_{\mathbb{P}^n}) \cong \mathcal{O}_Y \longrightarrow 0
\end{aligned}$$

which gives what we want. □

Remark 2.1.2. It is also a nice exercise to compare the degree, the arithmetic genus, the Hilbert function, and the Hilbert polynomial of two directly CI-linked (resp., G-linked) ideals. See, for instance, [42] for the complete intersection case and [72, Corollary 3.6] for the Gorenstein case.

Theorem 2.1.3. *Let $X, Y \subset \mathbb{P}^n$ be two equidimensional locally Cohen–Macaulay subschemes of the same dimension $d \geq 1$ directly G-linked by an AG subscheme $Z \subset \mathbb{P}^n$ with a minimal free R-resolution,*

$$0 \longrightarrow R(-t) \longrightarrow F_{n-d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow I(Z) \longrightarrow 0.$$

Then

$$M^{d-i+1}(Y) \cong M^i(X)^\vee(n+1-t) \quad \textit{for all} \quad 1 \leq i \leq d.$$

Proof. We consider a locally free resolution for the ideal sheaf $\mathcal{I}_X$of X as in Proposition 2.1.1,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_{n-d-1} \xrightarrow{f_{n-d-1}} \cdots \longrightarrow \mathcal{G}_2 \xrightarrow{f_2} \mathcal{G}_1 \xrightarrow{f_1} \mathcal{I}_X \longrightarrow 0, \tag{2.2}$$

where $\mathcal{G}$ is locally free and the $\mathcal{G}_i$ are free. Then by Proposition 2.1.1 we have a locally free resolution for the ideal sheaf $\mathcal{I}_Y$ of Y,

$$0 \longrightarrow \mathcal{G}_1^\vee(-t) \longrightarrow \mathcal{F}_1^\vee(-t) \oplus \mathcal{G}_2^\vee(-t) \stackrel{g_{n-d-1}}{\longrightarrow} \cdots \longrightarrow \mathcal{G}_{n-d-1}^\vee(-t) \oplus \mathcal{F}_{n-d-2}^\vee(-t) \\ \stackrel{g_2}{\longrightarrow} \mathcal{G}^\vee(-t) \oplus \mathcal{F}_{n-d-1}^\vee(-t) \longrightarrow \mathcal{I}_Y \longrightarrow 0. \qquad (2.3)$$

Set $\mathcal{H}_i := \operatorname{Ker}(f_{i-1})$ and $\mathcal{K}_i := \operatorname{Ker}(g_{n-d-i})$. Cutting (2.3) into short exact sequences and beginning from the left, one sees that

$$\begin{aligned} 0 = H_*^{n-d-2}(\mathcal{K}_2) &\cong \cdots \cong H_*^2(\mathcal{K}_{n-d-2}) \cong H_*^1(\mathcal{K}_{n-d-1}), \\ 0 = H_*^{n-d-1}(\mathcal{K}_2) &\cong \cdots \cong H_*^3(\mathcal{K}_{n-d-2}) \cong H_*^2(\mathcal{K}_{n-d-1}), \\ &\vdots \\ 0 = H_*^{n-2}(\mathcal{K}_2) &\cong \cdots \cong H_*^{d+2}(\mathcal{K}_{n-d-2}) \cong H_*^{d+1}(\mathcal{K}_{n-d-1}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} M^{d-i+1}(Y) &= H_*^{d-i+1}(\mathcal{I}_Y) \\ &\cong H^{d-i+1}(\mathcal{G}^\vee(-t)) && \text{by (2.3)} \\ &\cong H^{n-d+i-1}(\mathcal{G}(t-n-1))^* && \text{by Serre duality} \\ &\cong H^{n-d+i-2}(\mathcal{H}_{n-d-1}(t-n-1))^* && \text{by (2.2)} \\ &\vdots \\ &\cong H^{i+1}(\mathcal{H}_2(t-n-1))^* && \text{by (2.2)} \\ &\cong H^i(\mathcal{I}_X(t-n-1))^* && \text{by (2.2)} \\ &= (M^i)^\vee(X)(n+1-t). \end{aligned}$$

□

This last theorem says that the deficiency modules are invariant, up to duals and shifts and re-indexing, in a G-liaison class as well as in a CI-liaison class (see also [77]). In particular, an important consequence of Theorem 2.1.3 is that the property of being ACM is invariant in a G-liaison class and in a CI-liaison class, since this property depends only on the collection of deficiency modules being zero. In fact, we have the following corollary.

Corollary 2.1.4. *Let $X, Y \subset \mathbb{P}^n$ be two equidimensional locally Cohen–Macaulay subschemes of the same dimension $d \geq 1$. Assume that X and Y are directly G-linked by Z. Then X is ACM if and only if Y is ACM.*

Example 2.1.5. Let $C \subset \mathbb{P}^3$ be the rational quartic curve parameterized by (s^4, s^3t, st^3, t^4) and with homogeneous ideal $I(C) = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3, x_2^3 - x_1x_3^2, x_1^3 - x_0^2x_2)$. The curve C is contained in the complete intersection curve $X \subset \mathbb{P}^3$ defined by $I(X) = (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3) \subset I(C)$. It is easy to check that $I(X) : I(C) = (x_0, x_1) \cap (x_2, x_3)$. This implies that C is geometrically CI-linked to a pair $Y = L_1 \cup L_2$ of disjoint lines. Since a pair of disjoint lines is not

ACM (indeed, $M(Y) = \oplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_Y(t)) = H^1(\mathbb{P}^3, \mathcal{I}_Y) = K \neq 0$), we conclude that the rational quartic $C \subset \mathbb{P}^3$ is not ACM. Geometrically, it is clear that C is not ACM because it is obtained as a projection of the rational normal quartic from $\mathbb{P}^4$ to $\mathbb{P}^3$. Hence, C is not linearly normal or, equivalently, $H^1(\mathbb{P}^3, I_C(1)) \neq 0$.

Theorem 2.1.6 (Gaeta's theorem). *Let $V \subset \mathbb{P}^n$ be a codimension 2 subscheme defined by the maximal minors of a $t \times (t+1)$ homogeneous matrix $\mathcal{A}$. Then, V is CI-linked in a finite number of steps to a complete intersection, i.e., V is licci.*

Sketch of the proof. We CI-link V to a scheme V_1 by means of a complete intersection $X \subset \mathbb{P}^n$ defined by two minimal generators of $I(V)$. V_1 is standard determinantal since it is ACM of codimension 2. Gaeta proved that the Hilbert–Burch matrix $\mathcal{A}_1$ defining $I(V_1)$ is obtained from the Hilbert–Burch matrix $\mathcal{A}$ by deleting two columns and transposing:

$$\mathcal{A} = \begin{pmatrix} * & * & * & * & * & \bullet & \bullet \\ * & * & * & * & * & \bullet & \bullet \\ * & * & * & * & * & \bullet & \bullet \\ * & * & * & * & * & \bullet & \bullet \\ * & * & * & * & * & \bullet & \bullet \\ * & * & * & * & * & \bullet & \bullet \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}.$$

In this case, we can write $I(X) = (F_1, F_2)$, where F_1 is the determinant of the matrix obtained by deleting the last column of $\mathcal{A}$, and F_2 is the determinant of the matrix obtained by deleting the penultimate column of $\mathcal{A}$. Going on, in a finite number of steps, we reach a 1×2 matrix, i.e., a complete intersection. □

2.2 CI-liaison class of standard determinantal ideals

We have seen in Theorem 2.1.3 a necessary condition for two schemes to be in the same CI-liaison class or in the same G-liaison class, one would like to know if this condition is also a sufficient condition. In particular, since we know that a scheme $X \subset \mathbb{P}^n$ is ACM if and only if its deficiency modules vanish, we can ask the following question.

Question 2.2.1. *Do all the standard determinantal schemes belong to the same CI-liaison class (resp., G-liaison class)?*

Or, more generally, the following one.

Question 2.2.2. *Do all the ACM schemes belong to the same CI-liaison class* (*resp., G-liaison class*)*? Or, equivalently, is there only one CI-liaison* (*resp., G-liaison*) *class containing ACM schemes?*

We have seen that there is only one CI-liaison class containing ACM subschemes of codimension 2 and we also have the following result due to P. Schwartau [80].

Proposition 2.2.3. *All codimension c complete intersections $X \subset \mathbb{P}^n$ belong to the same CI-liaison class. More precisely, any two complete intersections of the same codimension are CI-linked in finitely many steps.*

Proof. The proof rests on the following observation: If $I(X_1) = (F_1, \dots, F_{c-1}, F)$ and $I(X_2) = (F_1, \dots, F_{c-1}, G)$ are two complete intersections of codimension c, then they are directly CI-linked by the complete intersection $I(X) = (F_1, \dots, F_{c-1}, FG)$. So, now starting with arbitrary complete intersections Y_1 and Y_2, one can produce a sequence of CI-links from Y_1 to Y_2 in c steps, just apply the observation to change one generator at a time. □

The goal of this section is to give, in the context of CI-liaison, a negative answer to Questions 2.2.1 and 2.2.2 in codimension ≥ 3. More precisely, we will prove the existence of infinitely many CI-liaison classes containing standard determinantal curves $C \subset \mathbb{P}^4$ and also the existence of infinitely many CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ (not necessarily standard). So, in the CI-liaison context Gaeta's theorem does not generalize well to subschemes $X \subset \mathbb{P}^n$ of higher codimension. Let us start by introducing some graded modules which are liaison invariants under CI-liaison but not under G-liaison (Theorems 2.2.6 and 2.2.8 and Example 2.2.16); and we will use them to prove the existence of infinitely many different CI-liaison classes containing standard determinantal curves $C \subset \mathbb{P}^4$ (Corollary 2.2.13 and Example 2.2.11).

Definition 2.2.4. Let $X \subset \mathbb{P}^n$ be a locally Cohen–Macaulay equidimensional subscheme. A graded R-module $C(X)$ which depends only on X is a *CI-liaison* (resp., *G-liaison*) *invariant* as an R-module (resp., K-module) if there exists a homogeneous R- (resp., K-) module isomorphism $C(X) \cong C(X')$ for any X' in the CI-liaison (resp., G-liaison) class of X.

By Theorem 2.1.3, for equidimensional locally Cohen–Macaulay subschemes $X \subset \mathbb{P}^n$, the ith modules of deficiency

$$M^i(X) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}_X(t)), \quad 1 \leq i \leq \dim(X),$$

are CI-liaison invariants (up to shifts and duals). Even more, they are G-liaison invariants. We will now describe other CI-liaison invariants which allow us to distinguish between many CI-liaison classes which cannot be distinguished by deficiency modules alone.

Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme, locally Cohen–Macaulay, equidimensional of $\dim n > 0$.[1] If X is ACM, all the CI-liaison invariants $M^i(X)$, $1 \le i \le \dim(X)$, vanish. Our first goal is to describe nontrivial CI-liaison invariants of ACM schemes. To this end, we consider a graded free R-resolution of $I = I(X)$,

$$\cdots \oplus_i R(-n_{2,i}) \longrightarrow \oplus_i R(-n_{1,i}) \longrightarrow I \longrightarrow 0. \tag{2.4}$$

We apply the contravariant functor $\mathcal{H}om(-, \mathcal{O}_X)$ to the sheafification of the exact sequence (2.4) and we obtain

$$0 \longrightarrow \mathcal{N}_X \longrightarrow \oplus_i \mathcal{O}_X(-n_{1,i}) \longrightarrow \oplus_i \mathcal{O}_X(-n_{2,i}).$$

We take cohomology ($H^n_* \mathcal{O}_X \cong H^{n+1}_{\mathfrak{m}}(R/I)$, $A = R/I(X)$); and we get a natural map,

$$\begin{aligned} \delta_X : H^n_* \mathcal{N}_X \longrightarrow\ & \mathrm{Hom}_R(I, H^{n+1}_{\mathfrak{m}}(A)) \\ \cong\ & \mathrm{Hom}_R(I, H^{n+2}_{\mathfrak{m}}(I)). \end{aligned}$$

This map δ_X plays an important role; in particular, its kernel and cokernel are CI-liaison invariants (see Theorem 2.2.6).

Remark 2.2.5. If I/I^2 is a free R/I-module, then δ_X is an isomorphism. In particular, if $X \subset \mathbb{P}^{n+c}$ is a global complete intersection, then δ_X is an isomorphism.

Theorem 2.2.6. *Let $X, X' \subset \mathbb{P}^{n+c}$ be ACM subschemes of dimension $n > 0$ directly CI-linked by a complete intersection $Y \subset \mathbb{P}^{n+c}$. Then*

(1) *as graded R-modules:*

$$\begin{aligned} H^i_* \mathcal{N}_X &\cong H^i_* \mathcal{N}_{X'} \quad \textit{for } 1 \le i \le n-1, \\ \ker(\delta_X) &\cong \ker(\delta_{X'}); \end{aligned}$$

(2) *as graded K-modules:*

$$\mathrm{Coker}(\delta_X) \cong \mathrm{Coker}(\delta_{X'});$$

(3) *moreover, if $Y \subset \mathbb{P}^{n+c}$ is a complete intersection of type $f_1, \ldots, f_c$, we have*

$$h^0 \mathcal{N}_X = h^0 \mathcal{N}_{X'} + \sum_{i=1}^{c} h^0(\mathcal{I}_{X'}(f_i)) - \sum_{i=1}^{c} h^0(\mathcal{I}_X(f_i)).$$

Proof. See [56, Theorem 6.1 and Proposition 9.20]. □

[1]Throughout this section we work with schemes of dimension $n > 0$. We want to point out that the results we give generalize to 0-dimensional schemes and we assume $n > 0$ to avoid technical complications.

As application, we get the following useful criterion to check if an ACM scheme is licci.

Corollary 2.2.7. *Let $X \subset \mathbb{P}^{n+c}$ be a closed subscheme of dimension $n > 0$. If X is licci, then*

(1) $H^i_*\mathcal{N}_X = 0$ *for* $1 \le i \le n-1$, *and*

(2) δ_X *is an isomorphism.*

Proof. This follows from Theorem 2.2.6 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+c}$, $H^i_*\mathcal{N}_Y = 0$ for $1 \le i \le n-1$ and δ_Y is an isomorphism (Remark 2.2.5). □

We will now restrict our attention to closed subschemes $X \subset \mathbb{P}^{n+3}$, $n > 0$, of codimension 3 and we will deduce from the previous results the CI-liaison invariance of the local cohomology groups

$$H^i_{\mathfrak{m}}(K_{R/I(X)} \otimes_R I(X)), \quad i = 0, \ldots, n,$$

where $I = I(X)$ is the homogeneous ideal of X and

$$K_{R/I(X)} = \operatorname{Ext}^3_R(R/I(X), R)(-n-4)$$

is the canonical module of X.

Indeed, let ${}_\mu \operatorname{Ext}^i_{\mathfrak{m}}(M, -)$ be the right-derived functor of ${}_\mu H^0_{\mathfrak{m}}(\operatorname{Hom}(M, -))$. Using basic facts on local cohomology, the spectral sequence relating local and global Ext

$$E_2^{pq} := H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}, \mathcal{G}), \tag{2.5}$$

and the spectral sequences (see [36, Exp. VI, Théorème 1.6])

$$E_2^{pq} :={}_\mu \operatorname{Ext}^p_R(M_1, H^q_{\mathfrak{m}}(M_2)) \Rightarrow{}_\mu \operatorname{Ext}^{p+q}_{\mathfrak{m}}(M_1, M_2), \tag{2.6}$$

$$E_2^{pq} :={}_\mu H^p_{\mathfrak{m}}(\operatorname{Ext}^q_R(M_1, M_2)) \Rightarrow{}_\mu \operatorname{Ext}^{p+q}_{\mathfrak{m}}(M_1, M_2), \tag{2.7}$$

we obtain the following theorem.

Theorem 2.2.8. *Let $X \subset \mathbb{P}^{n+3}$ be an ACM subscheme of codimension 3 ($n > 0$) and $K_X := \operatorname{Ext}^3_R(R/I(X), R)(-n-4)$ its canonical module. Then, we have*

(1) $H^{i+1}_*\mathcal{N}_X \cong H^i_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4)$, $0 \le i \le n-2$, *as graded R-modules;*

(2) *there exists an exact sequence*

$$\begin{aligned} 0 \to H^{n-1}_{\mathfrak{m}}&(K_X \otimes_R I(X))(n+4) \to H^n_*\mathcal{N}_X \\ &\xrightarrow{\delta_X} \operatorname{Hom}(I(X), H^{n+1}_{\mathfrak{m}}(R/I(X))) \\ &\to H^n_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4) \to 0. \end{aligned}$$

In particular,

(3) $H^i_{\mathfrak{m}}(K_X \otimes_R I(X))$ *are CI-liaison invariants as graded* R*- (resp.,* K*-) modules,* $0 \le i < n$ *(resp.,* $0 \le i \le n$*). Moreover, if* X *is locally Gorenstein, then*

$$H^i_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4) \cong H^{n-i}_{\mathfrak{m}}(K_X \otimes_R I(X))^{\vee}, \quad i = 0, \ldots, n,$$

as R*-modules.*

Proof. (1) Using the spectral sequence (2.6) and the fact that $H^q_{\mathfrak{m}}(I(X)) = 0$ for $q \le n+1$, we get that

$$\mathrm{Ext}^j_{\mathfrak{m}}(I(X), I(X)) = 0 \quad \text{for } j \le n+1. \tag{2.8}$$

Since X has codimension 3, i.e., $\mathrm{pd}_R\, I(X) = 2$, we also have $\mathrm{Ext}^j_R(I(X), I(X)) = 0$ for $j \ge 3$ and $\mathrm{Hom}_R(I(X), I(X)) = R$. Therefore, if $p+q < n+4$ and $M_1 = M_2 = I(X)$, the spectral sequence (2.7) consists of at most two nonvanishing terms. By (2.8), the spectral sequence (2.7) converges to zero, thus we have

$$H^i_{\mathfrak{m}}(\mathrm{Ext}^2_R(I(X), I(X))) \cong H^{i+2}_{\mathfrak{m}}(\mathrm{Ext}^1_R(I(X), I(X))).$$

Since $\mathcal{N}_X \cong \mathcal{E}xt^1(I(X), I(X))$, we get

$$\begin{aligned} H^{i+1}_*\mathcal{N}_X &\cong H^{i+1}_*(\mathcal{E}xt^1(I(X), I(X))) \\ &\cong H^{i+2}_{\mathfrak{m}}(\mathrm{Ext}^1_R(I(X), I(X))) \\ &\cong H^i_{\mathfrak{m}}(\mathrm{Ext}^2_R(I(X), I(X))) \\ &\cong H^i_{\mathfrak{m}}(\mathrm{Ext}^3_R(R/I(X), I(X))) \\ &\cong H^i_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4), \end{aligned}$$

which proves (1).

(2) By the spectral sequence (2.6),

$$\mathrm{Ext}^{n+2}_{\mathfrak{m}}(I(X), I(X)) \cong \mathrm{Hom}_R(I(X), H^{n+1}_{\mathfrak{m}}(R/I(X))).$$

Hence, using the spectral sequence (2.7), we obtain the exactness of

$$\begin{aligned} 0 &\to H^{n-1}_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4) \\ &\to H^n_*\mathcal{N}_X \xrightarrow{\delta_X} \mathrm{Hom}(I(X), H^{n+1}_{\mathfrak{m}}(R/I(X))) \\ &\to H^n_{\mathfrak{m}}(K_X \otimes_R I(X))(n+4) \to 0 \end{aligned} \tag{2.9}$$

where the zero to the right is by reasons of dimension.

(3) The CI-liaison invariance of $H^i_{\mathfrak{m}}(K_X \otimes_R I(X))$ immediately follows from (1), (2), and Theorem 2.2.6. Let us prove the duality. By [44], for any $x \in X$,

$(\mathcal{I}_X/\mathcal{I}_X^2 \otimes \omega_X)_x$ is Cohen–Macaulay. Hence, $\mathcal{E}xt^i(\mathcal{I}_X/\mathcal{I}_X^2 \otimes \omega_X, \omega_X) = 0$ for $i > 0$ by local Gorenstein duality and

$$\begin{aligned} H_{\mathfrak{m}}^{i+1}(\mathrm{Ext}_R^3(R/I(X), I(X))) &\cong H_*^i(\mathcal{E}xt^3_{\mathcal{O}_{\mathbb{P}^{n+3}}}(\mathcal{O}_X, \mathcal{I}_X)) \\ &\cong H_*^i(\mathcal{I}_X/\mathcal{I}_X^2 \otimes_{\mathcal{O}_X} \omega_X)(n+4) \\ &\cong \mathrm{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2 \otimes_{\mathcal{O}_X} \omega_X, \omega_X)^\vee(n+4) \\ &\cong H_*^{n-i}(\mathcal{N}_X)^\vee(n+4). \end{aligned}$$

Therefore, using the first isomorphism of this theorem, we get, for $2 \le i \le n-2$,

$$\begin{aligned} H_{\mathfrak{m}}^i(K_X \otimes_R I(X))(n+4) &\cong H_*^{i+1}(\mathcal{N}_X) \cong H_{\mathfrak{m}}^{n-i}(\mathrm{Ext}_R^3(R/I(X), I(X)))^\vee(n+4) \\ &\cong H_{\mathfrak{m}}^{n-i}(K_X(n+4) \otimes_R I(X))^\vee(n+4) \\ &\cong H_{\mathfrak{m}}^{n-i}(K_X \otimes_R I(X))^\vee. \end{aligned}$$

Moreover, by Gorenstein duality, we have

$$\begin{aligned} H_{\mathfrak{m}}^{i+1}(K_X \otimes I(X)/I(X)^2) &\cong \mathrm{Ext}^{n-i}_{R/I(X)}(K_X \otimes I(X)/I(X)^2, K_X)^\vee \\ &\cong \mathrm{Ext}^{n-i}_{R/I(X)}(I(X)/I(X)^2, R/I(X))^\vee \end{aligned}$$

for $i \ge -1$, where the last isomorphism follows from [43] and the fact that X is locally Gorenstein. Since $H_*^n(\mathcal{N}_X) \cong \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$, we conclude by comparing the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^n_{R/I(X)}(I(X)/I(X)^2, R/I(X)) \to \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X) \\ &\to \mathrm{Hom}_{R/I(X)}(I(X)/I(X)^2, H_{\mathfrak{m}}^{n+1}(R/I(X))) \\ &\to \mathrm{Ext}^{n+1}_{R/I(X)}(I(X)/I(X)^2, R/I(X)) \to 0 \end{aligned}$$

with the exact sequence (2.9). □

As an application we get another useful criterion to check if a codimension 3 ACM subscheme X of $\mathbb{P}^n$ is licci.

Corollary 2.2.9. *Let $X \subset \mathbb{P}^{n+3}$ be a closed subscheme of dimension $n > 0$. If X is licci, then $H_{\mathfrak{m}}^i(K_X \otimes_R I(X)) = 0$ for $0 \le i \le n$.*

Proof. This follows from Theorem 2.2.8 and the fact that for complete intersections $Y \subset \mathbb{P}^{n+3}$, $H_{\mathfrak{m}}^i(K_{R/I(Y)} \otimes_R I(Y)) = 0$ for $0 \le i \le n$. □

We are led to pose the following question which, to our knowledge, is still open.

Question 2.2.10. *The question is whether the converse of Corollary 2.2.9 is true; i.e., is a codimension 3 ACM scheme $X \subset \mathbb{P}^{n+3}$ licci if $H_{\mathfrak{m}}^i(K \otimes_R I(X)) = 0$ for $0 \le i \le n$?*

Now, we will illustrate by means of an example how to use Theorem 2.2.8.

Example 2.2.11. Let $C \subset \mathbb{P}^4$ be a local complete intersection curve of degree d and arithmetic genus g with an *almost linear* resolution,

$$0 \to R(-s-3)^a \to R(-s-2)^b \to R(-s-1)^{c_1} \oplus R(-s)^{c_0} \to I(C) \to 0.$$

If $d+g-1-ac_0 \neq 0$, then C is not licci.

Idea of the proof. We compute the dimension,

$$l(C)_\mu := \dim_{\mu+5} H^0_{\mathfrak{m}}(K_{R/I(C)} \otimes_R I(C)),$$

of the CI-liaison invariants ${}_{\mu+5}H^0_{\mathfrak{m}}(K_{R/I(C)} \otimes_R I)$. The exact sequence and the duality of Theorem 2.2.8 give us (small letters mean dimension)

$$l(C)_\mu - l(C)_{-\mu-5} = h^1\mathcal{N}_C(\mu) -_\mu \hom_R(I(C), H^2_{\mathfrak{m}}(R/I(C))).$$

Since ${}_{-2}\hom_R(I, H^2_{\mathfrak{m}}(R/I(C))) = ac_0$ and $h^1\mathcal{N}_C(-2) = -\chi\mathcal{N}_C(-2) = d + g - 1$ (Riemann–Roch's theorem), we obtain

$$\begin{aligned} l(C)_{-2} - l(C)_{-3} &= h^1\mathcal{N}_C(-2) -_{-2} \hom_R(I(C), H^2_{\mathfrak{m}}(R/I(C))) \\ &= d+g-1-ac_0. \end{aligned}$$

Therefore, by Corollary 2.2.9, if $d+g-1-ac_0 \neq 0$, then C is not licci. □

Remark 2.2.12.

(1) The only smooth connected curve in $\mathbb{P}^4$ with a linear resolution ($c_0 = 0$) which is licci is a line.

(2) The smooth rational normal quartic $C \subset \mathbb{P}^4$ is not licci. Indeed, $(a, b, c_1, c_0, s) = (3, 8, 6, 0, 1)$ and $d+g-1-ac_0 = 3 \neq 0$.

(3) As examples of smooth ACM curves $C \subset \mathbb{P}^4$ with an almost linear resolution we have the following:

(3i) Let $S \subset \mathbb{P}^4$ be a Castelnuovo surface, consider a general curve $C \in |5E_0 - 2E_1 - \sum_{i=2}^{8} E_i|$, then $\deg(C) = 9$, $g(C) = 5$, and $I(C)$ has a minimal free R-resolution,

$$0 \to R(-5)^5 \to R(-4)^{12} \to R(-3)^7 \oplus R(-2) \to I(C) \to 0.$$

(3ii) Let $L \subset \mathbb{P}^4$ be a line and $Q_1, Q_2, Q_3 \subset \mathbb{P}^4$ three general hyperquadrics containing L. We denote by $X \subset \mathbb{P}^4$ the ACM curve CI-linked to L by $Q_1 \cap Q_2 \cap Q_3$. Then $\deg(X) = 7$, $g(X) = 3$, and $I(X)$ has a minimal free R-resolution,

$$0 \to R(-5)^3 \to R(-4)^6 \to R(-3) \oplus R(-2)^3 \to I(X) \to 0.$$

It would be nice to describe which invariants (a, b, c_1, c_0, s) (or which d and g) occur in Example 2.2.11.

We will now deduce the existence of infinitely many different CI-liaison classes containing standard determinantal curves $C \subset \mathbb{P}^4$, and also the existence of infinitely many CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ (not necessarily standard).

Corollary 2.2.13. *Let $C_t \subset \mathbb{P}^4$ be a local complete intersection ACM curve with a linear resolution,*

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0.$$

For $t \neq q$, C_t and C_q belong to different CI-liaison classes.

Proof. Using the minimal free R-resolution of $I(C_t)$, we compute the Hilbert polynomial of $R/I(C_t)$ and we obtain

$$p_{C_t}(x) = \left(\binom{t+3}{4} - \binom{t+2}{4}\right) x - (t-1)d(C_t) + \binom{t+3}{4}.$$

Hence, we have

$$d(C_t) = \binom{t+3}{4} - \binom{t+2}{4},$$
$$p_a(C_t) = (t-1)d(C_t) + 1 - \binom{t+3}{4}.$$

We can easily check that $d(C_t) + p_a(C_t) - 1 \neq d(C_q) + p_a(C_q) - 1$ for $t \neq q$. Therefore, by Example 2.2.11, C_t and C_q belong to different liaison classes provided $t \neq q$. □

Remark 2.2.14. Since standard determinantal curves $X \subset \mathbb{P}^4$, defined by the maximal minors of a $t \times (t+2)$ matrix $\mathcal{A}$ with linear entries, have a linear resolution, we deduce from Corollary 2.2.13 the existence of infinitely many CI-liaison classes containing standard determinantal curves of codimension 3.

Remark 2.2.15. Corollary 2.2.13 and Remark 2.2.14 show that in the context of CI-liaison, Gaeta's theorem does not generalize either to ACM subschemes $X \subset \mathbb{P}^n$ of higher codimension or to standard determinantal subschemes $X \subset \mathbb{P}^n$ of higher codimension. In the next section, we will try to convince the reader that G-liaison is a more natural approach if we want to carry out a program in higher codimension.

In Theorem 2.2.8, we have seen that, in codimension 3, the graded modules $H^i_{\mathfrak{m}}(K_X \otimes_R I(X))$ are CI-liaison invariants. We would like to know if they are also G-liaison invariants. Unfortunately, the following example shows that $H^i_{\mathfrak{m}}(K_X \otimes_R I(X))$ are not G-liaison invariants. Indeed, we have the following example.

Example 2.2.16. Let $D_t \subset \mathbb{P}^4$ be an ACM curve defined by the maximal minors of a $t \times (t+2)$ matrix with linear entries. D_t has a linear resolution . According to the proof of Example 2.2.11, $H^0_{\mathfrak{m}}(K_{D_t} \otimes_R I(D_t))$ changes when t varies, and it follows from Theorem 2.3.1 that D_t is glicci. Therefore, $H^0_{\mathfrak{m}}(K_X \otimes_R I(X))$ is not a G-liaison invariant.

As another example about the existence of infinitely many different CI-liaison classes containing ACM curves $C \subset \mathbb{P}^4$ we have the following one.

Example 2.2.17. Let $S \subset \mathbb{P}^4$ be a Castelnuovo (resp., Bordiga) surface and let $C \subset S$ be a rational, normal quartic. Consider an effective divisor $C_t \in |C + tH|$, where H is a hyperplane section of S and $0 \leq t \in \mathbb{Z}$. It holds that

- C_t is not licci, for all $t \geq 0$;
- C_t and $C_{t'}$ belong to different CI-liaison classes provided $0 \leq t < t'$.

By Theorem 1.3.12, all these examples of ACM curves $C_t = C + tH \subset S \subset \mathbb{P}^4$, which belong to different CI-liaison classes, belong to the same G-liaison class. So the situation drastically changes when we link by means of AG schemes instead of complete intersections. G-liaison is in many ways more natural than CI-liaison, and using the fact that, roughly speaking, G-liaison is a theory about generalized divisors on ACM schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on a complete intersection, we will generalize Gaeta's theorem to standard determinantal schemes of arbitrary codimension.

2.3 G-liaison class of standard determinantal ideals

The goal of this section is to prove that standard determinantal schemes are glicci; i.e., they belong to the G-liaison class of a complete intersection.

Theorem 2.3.1. *Let $V \subset \mathbb{P}^n$ be a standard determinantal scheme of codimension c defined by the maximal minors of a $t \times (t + c - 1)$ homogeneous matrix $\mathcal{A}$. Then, V is glicci.*

Proof. The proof is rather technical and the main idea is the following one.

We denote by $\mathcal{B}$ the matrix obtained by deleting a "suitable" column of $\mathcal{A}$ and we call X the subscheme defined by the maximal minors of $\mathcal{B}$. ("Suitable" means that $\operatorname{codim}(X) = c - 1$. First take, if necessary, a general linear combination of the rows and columns of $\mathcal{A}$.) We denote by $\mathcal{A}'$ the matrix obtained by deleting a "suitable" row of $\mathcal{B}$ and we call V' the subscheme defined by the maximal minors of $\mathcal{A}'$. ("Suitable" means that $\operatorname{codim}(V') = c$. First take, if necessary, a general linear combination of the rows and columns of $\mathcal{B}$.)

$$\mathcal{A} = \begin{pmatrix} * & * & * & * & * & * & * & \bullet \\ * & * & * & * & * & * & * & \bullet \\ * & * & * & * & * & * & * & \bullet \\ * & * & * & * & * & * & * & \bullet \\ \square & \square & \square & \square & \square & \square & \square & \bullet \end{pmatrix} \qquad I(\mathcal{A}) = I(V), \quad \operatorname{codim}(V) = c,$$

$$\mathcal{B} = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ \square & \square & \square & \square & \square & \square & \square \end{pmatrix} \quad I(\mathcal{B}) = I(X), \quad \operatorname{codim}(X) = c-1,$$

$$\mathcal{A}_1 = \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{pmatrix} \quad I(\mathcal{A}_1) = I(V'), \quad \operatorname{codim}(V') = c.$$

We consider V and V' as divisors on X, we show that V and V' are G-bilinked. Indeed, we denote by H the hyperplane section of X and we prove that $V' \in |V + tH|$ for a suitable $t \in \mathbb{Z}$. Therefore, by Theorem 1.3.12, V and V' are G-bilinked. Hence, in $2t-2$ steps we reach a scheme defined by a $1 \times c$ matrix; i.e., we arrive at a complete intersection.

As we see the proof is essentially an algorithm describing how the required links can be achieved. Now, we outline the steps of this algorithm and we refer the reader to [56, Theorem 3.6] for the complete proof.

Let $I \subset R$ be a standard determinantal ideal of codimension $c+1$ and let $\mathcal{A}$ be a $t \times (t+c)$ homogeneous matrix associated with the ideal I, so $I = I_t(\mathcal{A})$. If $t = 1$, then I is a complete intersection ideal and there is nothing to prove. Assume $t > 1$, then our assertion follows by induction on t if we have shown that I is evenly G-linked to a standard determinantal ideal I' generated by the maximal minors of a $(t-1) \times (t+c-1)$ matrix $\mathcal{A}'$. Actually we will see that $\mathcal{A}'$ can be chosen as the matrix obtained by deleting a suitable column and a suitable row of the matrix $\mathcal{A}$ and that I and I' are G-linked in two steps (i.e., G-bilinked). In order to do that, we proceed in several steps and we conclude that V is glicci.

Step 1. Let $\mathcal{B}$ be the $t \times (t+c-1)$ matrix obtained by deleting the last column of $\mathcal{A}$. Then the ideal $I(\mathcal{B})$ has codimension c; i.e., it is a standard determinantal ideal.

Possibly after elementary row operations we may assume that the maximal minors of the matrix $\mathcal{A}'$ obtained by deleting the last row of $\mathcal{B}$ generate an ideal $I' = I(\mathcal{A}')$ of maximal codimension $c+1$.

Step 2. Possibly after elementary column operations we may assume that the maximal minors of the matrix $\mathcal{A}_1$ consisting of the first $t-1$ columns of the matrix $\mathcal{A}$ generate an ideal of maximal codimension t. Put $J = I(\mathcal{A}_1)$. Let d be the determinant of the matrix which consists of the first $t-1$ and the last column of the matrix $\mathcal{A}$. Then one can show that

(i) $I(\mathcal{B}) : d = I(\mathcal{B})$,
(ii) $I = (I(\mathcal{B}) + dR) : J$,
(iii) $I(\mathcal{B}) + dJ^{c-1}$ is a Gorenstein ideal of codimension $c+1$, and
(iv) $\deg I(\mathcal{B}) + dJ^{c-1} = \deg d \cdot \deg(I(\mathcal{B}) + J^{c-1})$.

Step 3. Consider for $i = 0, 1, \ldots, c$ the ideals $I(\mathcal{B}) + J^i$. They are Cohen–Macaulay ideals of degree

$$\deg(I(\mathcal{B}) + J^i) = i(\deg d \cdot \deg I(\mathcal{B}) - \deg I).$$

Step 4. Comparing degrees, it is now not difficult to check that

$$(I(\mathcal{B}) + dJ^{c-1}) : I = I(\mathcal{B}) + J^c.$$

Step 5. Let d' be the determinant of the matrix consisting of the first $t-1$ columns of $\mathcal{A}'$. Then, similarly as above, $I(\mathcal{B}) + d'J^{c-1}$ is a Gorenstein ideal of codimension $c+1$ and

$$(I(\mathcal{B}) + d'J^{c-1}) : I' = I(\mathcal{B}) + J^c.$$

Step 6. Step 5 says that the ideal I' is directly G-linked to $I(\mathcal{B}) + J^c$, while Step 4 says that the ideal I is directly G-linked to $I(\mathcal{B}) + J^c$ which proves what we want. □

Example 2.3.2. We consider a rational normal curve $X \subset \mathbb{P}^n$ and we will prove that X is glicci but not licci. It is well known that after a change of coordinates, we may assume that the homogeneous ideal of $I(X)$ can be generated by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}.$$

Hence, X is standard determinantal and therefore glicci by Theorem 2.3.1. On the other hand, the curve X has a linear free R-resolution,

$$0 \longrightarrow R(-n)^{b_{n-1}} \longrightarrow \cdots \longrightarrow R(-3)^{b_2} \longrightarrow R(-2)^{b_1} \longrightarrow I(X) \longrightarrow 0,$$

where $b_i = i\binom{n}{i+1}$. Hence, Example 2.2.11 implies that for $n = 4$, the curve X is not licci; i.e., X is not in the CI-liaison class of a complete intersection.

Remark 2.3.3. The way that Gaeta's theorem is usually stated is that, if $X \subset \mathbb{P}^n$ is a codimension 2 subscheme, then X is ACM if and only if X is licci. In higher codimension, we know that

$$\text{standard determinantal} \Rightarrow \text{glicci} \Rightarrow \text{ACM}.$$

The first converse is certainly false. The second is still an open problem.

This page intentionally left blank

Chapter 3

Multiplicity Conjecture for Standard Determinantal Ideals

Let $I \subset R = K[x_1, \ldots, x_n]$ be a graded ideal of arbitrary codimension c. Consider the minimal graded free R-resolution of R/I,

$$0 \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j \in \mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where, as usual, we denote by $\beta_{i,j}(R/I) = \dim \operatorname{Tor}_i^R(R/I, K)_j$ the (i,j)th graded Betti number of R/I and by $\beta_i(R/I) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(R/I)$ the ith total Betti number of R/I.

Many important numerical invariants of I and the associated scheme can be read off from the graded minimal free R-resolution of R/I. For instance, the Hilbert polynomial, the multiplicity of I, etc. We know well that $c \leq p$ and equality holds if and only if R/I is Cohen–Macaulay. We define

$$m_i(I) = \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$$

to be the minimum degree shift at the ith step and

$$M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$$

to be the maximum degree shift at the ith step. We will simply write m_i and M_i when there is no confusion. If R/I is Cohen–Macaulay and has a *pure resolution*, i.e., $m_i = M_i$ for all i, $1 \leq i \leq c$, then C. Huneke and M. Miller showed in [51] the following beautiful formula for the multiplicity of R/I:

$$e(R/I) = \frac{\prod_{i=1}^{c} m_i}{c!}.$$

Example 3.0.1. Let $X \subset \mathbb{P}^d$ be the rational normal curve defined as the image of the map

$$\mu_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d, \quad [a:b] \mapsto [a^d : a^{d-1}b : \cdots : ab^{d-1} : b^d].$$

It is well known that the homogeneous ideal of $I(X)$ can be generated by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

Hence, X is standard determinantal and the Eagon–Northcott complex gives us the minimal linear free R-resolution,

$$0 \longrightarrow R(-d)^{b_{d-1}} \longrightarrow \cdots \longrightarrow R(-3)^{b_2} \longrightarrow R(-2)^{b_1} \longrightarrow I(X) \longrightarrow 0,$$

where $b_i = i\binom{n}{i+1}$ and

$$e(R/I(X)) = \frac{\prod_{i=1}^{d-1}(i+1)}{(d-1)!} = d = \deg(X).$$

Generalizing their result, J. Herzog, C. Huneke, and H. Srinivasan [46] made the following conjecture (*multiplicity conjecture*) which bounds the multiplicity $e(R/I)$ of a homogeneous ideal $I \subset R$ in terms of the maximum and minimum shifts in its graded minimal free R-resolution.

Conjecture 3.0.2. *Let $I \subset R$ be a graded ideal of codimension c. If R/I is Cohen–Macaulay, then*

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}.$$

Conjecture 3.0.2 has been extensively studied, and partial results have been obtained. It turns out to be true for the following types of ideals:

- complete intersections [46],
- powers of complete intersection ideals [37],
- perfect ideals with a pure resolution [51],
- perfect ideals with a quasi-pure resolution (i.e., $m_i \geq M_{i-1}$) [46],
- perfect ideals of codimension 2 [46],
- Gorenstein ideals of codimension 3 [67],
- perfect stable monomial ideals [46],
- Perfect square free strongly stable monomial ideals [46],
- 0-dimensional subschemes Y that are residual to a zero scheme Z of certain type [29].

The goal of this chapter is to establish the multiplicity conjecture (Conjecture 3.0.2) for K-algebras $K[x_1, \dots, x_n]/I$, where I is a standard determinantal ideal of arbitrary codimension c, i.e., an ideal generated by the maximal minors of a $t \times (t+c-1)$ homogeneous polynomial matrix.

In Section 3.1, we prove the multiplicity conjecture for perfect ideals of codimension 2 (or, equivalently, standard determinantal ideals of codimension 2) and we give a characterization for the sharpness of the multiplicity conjecture in the codimension 2 case. In Section 3.2, we determine the minimal and maximal shifts in the graded minimal free R-resolution of R/I, where I is a standard determinantal ideal of codimension c, in terms of the degree matrix $\mathcal{U}$ of $\mathcal{A}$, $\mathcal{A}$ is the homogeneous matrix associated with I; and we state some technical lemmas used in the inductive process of the proof of our main theorem (cf. Theorem 3.2.6). The heart of Section 3.2 is completely devoted to proving Conjecture 3.0.2 for standard determinantal ideals I of arbitrary codimension. To prove it we use induction on the codimension c of I and, for any c, induction on the size t of the homogeneous $t \times (t+c-1)$ matrix whose maximal minors generate I by successively deleting columns and rows of the largest possible degree when we prove the lower bound, and of the smallest possible degree, when we prove the upper bound. In the last part of Section 3.2, we answer a question stated by T. Römer in [78], and we prove that the ith total Betti number $\beta_i(R/I)$ of a standard determinantal ideal I can be bounded above by a function of the maximal shifts in the minimal graded free R-resolution of R/I as well as bounded below by a function of both the maximal and the minimal shifts.

The following remark is very useful.

Remark 3.0.3. In order to establish the multiplicity conjecture for a Cohen–Macaulay ideal $I \subset R$, it is enough to prove the multiplicity conjecture for the Artinian reduction J of I. In fact, I and its Artinian reduction J have the same multiplicity and the same graded Betti numbers.

3.1 The multiplicity conjecture for Cohen–Macaulay codimension 2 ideals

In this section, we prove the multiplicity conjecture for codimension 2 perfect ideals. The result was first proved by J. Herzog and H. Srinivasan in [46] where they also prove the multiplicity conjecture for perfect ideals with a quasi-pure resolution.

To begin with, we consider a perfect graded ideal $I \subset R$ of codimension 2. According to Hilbert–Burch theorem (see Theorem 1.2.18) there is an $r \times (r+1)$ homogeneous matrix $\mathcal{A} = (g_{ij})$, the Hilbert–Burch matrix, whose maximal minors generate I, giving rise to a minimal free R-resolution of I,

$$0 \longrightarrow \oplus_{i=1}^{m-1} R(-b_i) \xrightarrow{\mathcal{A}} \oplus_{j=1}^{m} R(-a_j) \longrightarrow I \longrightarrow 0.$$

Without loss of generality, we may assume $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \leq b_2 \leq \cdots \leq b_{m-1}$. We set $u_{ij} = \deg g_{ij} = b_i - a_j$ for all i and j and, as usual, we call $\mathcal{U} = (u_{ij})$ the degree matrix of $\mathcal{A}$. Set $u_i = u_{ii}$ and $v_i = u_{ii+1}$. The following relations were observed, for example, in [48]:

(a) $u_i \geq v_i \geq 0$ for $i = 1, \ldots, m-1$;
(b) $u_{i+1} \geq v_i$ for $i = 1, \ldots, m-2$;
(c) $a_1 = v_1 + \cdots + v_{m-1}$ and $a_m = u_1 + \cdots + u_{m-1}$;
(d) $b_1 = v_1 + \cdots + v_{m-1} + u_1$ and $b_{m-1} = u_1 + \cdots + u_{m-1} + v_{m-1}$;
(e) $e(R/I) = \sum_{i=1}^{m-1} u_i(v_i + \cdots + v_{m-1})$.

We can easily check that the integers u_i and v_i determine the whole degree matrix $\mathcal{U}$, and conversely any two sequences of integers u_i and v_i satisfying (a) and (b) arise from a graded perfect ideal of codimension 2.

Theorem 3.1.1. *Let R/I be a graded Cohen–Macaulay algebra of codimension 2. Then it holds that*

$$\frac{m_1 m_2}{2!} \leq e(R/I) \leq \frac{M_1 M_2}{2!}.$$

Moreover, $\frac{m_1 m_2}{2!} = e(R/I)$ if and only if R/I has a pure resolution if and only if $e(R/I) = \frac{M_1 M_2}{2!}$.

Proof. We consider a minimal free R-resolution of I,

$$0 \longrightarrow \oplus_{i=1}^{m-1} R(-b_i) \xrightarrow{\mathcal{A}} \oplus_{j=1}^{m} R(-a_j) \longrightarrow I \longrightarrow 0.$$

By hypothesis, $a_1 \leq a_2 \leq \cdots \leq a_m$ and $b_1 \leq b_2 \leq \cdots \leq b_{m-1}$. So, we have $m_1 = a_1$, $m_2 = b_1$, $M_1 = a_m$, and $M_2 = b_{m-1}$ and we have to prove that

$$a_1 b_1 \leq 2e(R/I) \leq a_m b_{m-1}.$$

Let us start with the lower equality. We want to show that

$$a_1 b_1 = (v_1 + \cdots + v_{m-1})(v_1 + \cdots + v_{m-1} + u_1) \leq \sum_{i=1}^{m-1} 2u_i(v_i + \cdots + v_{m-1}) = 2e(R/I).$$

Since

$$2u_1(v_1 + \cdots + v_{m-1}) \geq u_1(v_1 + \cdots + v_{m-1}) + v_1(v_1 + \cdots + v_{m-1})$$

and $2u_i \geq v_{i-1} + v_i$, it suffices to show that

$$\begin{aligned}\sum_{i=2}^{m-1} &(v_{i-1} + v_i)(v_i + \cdots + v_{m-1}) \\ &= (v_1 + \cdots + v_{m-1})(v_2 + \cdots + v_{m-1}).\end{aligned} \tag{3.1}$$

The right-hand side of (3.1) may be written as

$$\begin{aligned}(v_1 + \cdots + v_{m-1})&(v_2 + \cdots + v_{m-1}) \\ &= (v_1 + v_2)(v_2 + \cdots + v_{m-1}) + (v_2 + \cdots + v_{m-1})(v_3 + \cdots + v_{m-1}).\end{aligned}$$

Substituting in (3.1) and cancelling the first summand of both sides of the equality, one obtains

$$\sum_{i=3}^{m-1} (v_{i-1} + v_i)(v_i + \cdots + v_{m-1}) = (v_2 + \cdots + v_{m-1})(v_3 + \cdots + v_{m-1})$$

which follows by induction.

In order to prove the upper bound for the multiplicity, we have to show that

$$\begin{aligned}2e(R/I) = \sum_{i=1}^{m-1} 2u_i(v_i + \cdots + v_{m-1}) &= \sum_{i=1}^{m-1} 2v_i(u_1 + \cdots + u_i) \\ &\leq (u_1 + \cdots + u_{m-1})(u_1 + \cdots + u_{m-1} + v_{m-1}) = a_m b_{m-1}.\end{aligned}$$

Observing that

$$2v_{m-1}(u_1 + \cdots + u_{m-1}) \leq v_{m-1}(u_1 + \cdots + u_{m-1}) + u_{m-1}(u_1 + \cdots + u_{m-1}),$$

and that $2v_i \leq u_i + u_{i+1}$ for $i = 1, \ldots, m-2$, it remains to show that

$$\begin{aligned}\sum_{i=1}^{m-2} &(u_1 + \cdots + u_i)(u_i + u_{i+1}) \\ &= (u_1 + \cdots + u_{m-1})(u_1 + \cdots + u_{m-2}).\end{aligned} \tag{3.2}$$

The right-hand side of (3.2) may be written as

$$(u_{m-2} + u_{m-1})(u_1 + \cdots + u_{m-2}) + (u_1 + \cdots + u_{m-2})(u_1 + \cdots + u_{m-3}).$$

The first of these summands cancels against the last summand of the left side of (3.2), and we obtain

$$\sum_{i=1}^{m-3} (u_1 + \cdots + u_i)(u_i + u_{i+1}) = (u_1 + \cdots + u_{m-2})(u_1 + \cdots + u_{m-3}).$$

Hence the assertion follows by induction.

Finally, we observe that the lower inequality is an equality if and only if $u_1 = v_1$ and $2u_i = v_{i-1} + v_i$ for $i = 2, \ldots, m-1$ and the upper inequality turns out to be an equality if and only if $v_{m-1} = u_{m-1}$ and $2v_i = u_i + u_{i+1}$ for $i = 2, \ldots, m-2$. Therefore, we have strict inequalities unless I has a pure resolution. □

We end this section with the following example.

Example 3.1.2. We consider the ideal $I = (z^5 - 2yz^2t^2 + xt^4 + y^2u^3 - xzu^3, x^4y^2 - x^5z + z^3t^3 - yt^5 - yz^2u^3 + xt^2u^3, x^4yz^2 - x^5t^2 + z^2t^5 - z^4u^3 - yt^3u^3 + xu^6, x^4z^3 - x^4yt^2 + t^7 - z^2t^2u^3 - zt^3u^3 + yu^6) \subset K[x, y, z, t, u]$. I is a codimension 2 perfect ideal generated by the maximal minors of the homogeneous matrix

$$\mathcal{A} = \begin{pmatrix} x & y & z^2 & t^3 \\ y & z & t^2 & u^3 \\ z^2 & t^2 & u^3 & x^4 \end{pmatrix}.$$

The degree matrix is

$$\mathcal{U} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 4 \end{pmatrix}$$

and I has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-8)^2 \oplus R(-9) \longrightarrow R(-5) \oplus R(-6) \oplus R(-7)^2 \longrightarrow I \longrightarrow 0.$$

So, we have

$$m_1(I) = 5, \quad M_1(I) = 7, \quad m_2(I) = 8, \quad \text{and} \quad M_2(I) = 9.$$

Moreover, $e(R/I) = 25$ and we can easily check that

$$\frac{m_1(I)m_2(I)}{2!} = 20 \le e(R/I) = 25 \le \frac{M_1M_2}{2!} = \frac{63}{2}.$$

3.2 The multiplicity conjecture for standard determinantal ideals

We now turn our attention to standard determinantal ideals I of arbitrary codimension c, and we prove the multiplicity conjecture for standard determinantal ideals I using induction on the codimension c of I.

Let $I \subset R$ be a standard determinantal ideal of codimension c generated by the maximal minors of a $t \times (t + c - 1)$ matrix $\mathcal{A} = (f_{ij})_{i=1,\dots,t}^{j=1,\dots,t+c-1}$ where $f_{ij} \in K[x_1, \dots, x_n]$ are homogeneous polynomials of degree $a_j - b_i$. The matrix $\mathcal{A}$ defines a degree 0 map

$$F = \oplus_{i=1}^{t} R(b_i) \xrightarrow{\mathcal{A}} G = \oplus_{j=1}^{t+c-1} R(a_j),$$
$$v \mapsto \mathcal{A} \cdot v^t,$$

where $v = (v_1, \dots, v_t) \in F$ and we assume, without loss of generality, that $\mathcal{A}$ is minimal; i.e., $f_{ij} = 0$ for all i, j with $b_i = a_j$. If we let $u_{i,j} = a_j - b_i$ for all

$j = 1, \ldots, t+c-1$ and $i = 1, \ldots, t$, the matrix $\mathcal{U} = (u_{i,j})_{i=1,\ldots,t}^{j=1,\ldots,t+c-1}$ is called the *degree matrix* associated with I. By re-ordering degrees, if necessary, we may also assume that $b_1 \geq \cdots \geq b_t$ and $a_1 \leq a_2 \leq \cdots \leq a_{t+c-1}$. In particular, we have

$$u_{i,j} \leq u_{i+1,j} \quad \text{and} \quad u_{i,j} \leq u_{i,j+1} \quad \text{for all } i, j. \tag{3.3}$$

Note that the degree matrix $\mathcal{U}$ is completely determined by $u_{1,1}$, $u_{1,2}$, $\ldots$, $u_{1,c}$, $u_{2,2}$, $u_{2,3}$, $\ldots$, $u_{2,c+1}$, $\ldots$, $u_{t,t}$, $u_{t,t+1}$, $\ldots$, $u_{t,c+t-1}$ because of the identity $u_{i,j} + u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} = 0$ for all i, j. Moreover, the graded Betti numbers in the minimal free R-resolution of $R/I(\mathcal{A})$ depend only upon the integers

$$\{u_{i,j}\}_{1 \leq i \leq t}^{i \leq j \leq c+i-1} \subset \{u_{i,j}\}_{i=1,\ldots,t}^{j=1,\ldots,t+c-1}$$

as described below.

Proposition 3.2.1. *Let $I \subset R$ be a standard determinantal ideal of codimension c with degree matrix $\mathcal{U} = (u_{i,j})_{i=1,\ldots,t}^{j=1,\ldots,t+c-1}$ as above. Then we have*

(1) $m_i = u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}$ *for* $1 \leq i \leq c$,

(2) $M_i = u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t,t+c-i} + u_{t,t+c-i+1} + u_{t,t+c-i+2} + \cdots + u_{t,t+c-1}$ *for* $1 \leq i \leq c$,

(3) $\beta_i(R/I) = \binom{t+c-1}{t+i-1}\binom{t+i-2}{i-1}$ *for* $1 \leq i \leq c$.

Proof. We denote by $\varphi : F \longrightarrow G$ the morphism of free graded R-modules of rank t and $t + c - 1$, defined by the homogeneous matrix $\mathcal{A}$ associated with I. The Eagon–Northcott complex $\mathcal{D}_0(\varphi^*)$,

$$0 \longrightarrow \wedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \wedge^t F \longrightarrow \wedge^{t+c-2} G^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \cdots$$
$$\longrightarrow \wedge^t G^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

gives us a graded minimal free R-resolution of R/I (See Proposition 1.2.16 (2)). Now the result follows after a straightforward computation taking into account that

$$\wedge^t F = R\left(\sum_{i=1}^{t} b_i\right),$$

$$S_a(F) = \bigoplus_{1 \leq i_1 \leq \cdots \leq i_a \leq t} R\left(\sum_{j=1}^{a} b_{i_j}\right),$$

$$G^* = \bigoplus_{j=1}^{t+c-1} R(-a_j),$$

$$\wedge^b G^* = \bigoplus_{1 \leq i_1 < \cdots < i_b \leq t+c-1} R\left(-\sum_{j=1}^{b} a_{i_j}\right). \qquad \square$$

We will now fix the notation and prove the technical lemmas needed in the induction process that we will use later for proving the multiplicity conjecture for standard determinantal ideals of arbitrary codimension.

Lemma-Definition 3.2.2. Let $0 \neq J \subset I \subset R$ be homogeneous ideals such that $\operatorname{codim} I = \operatorname{codim} J + 1$ and R/J is Cohen–Macaulay. Let $f \in R$ be a homogeneous element of degree d such that $J : f = J$. Then the ideal $\overline{I} := J + fI$ satisfies $\operatorname{codim} \overline{I} = \operatorname{codim} I$ and

$$H^i_{\mathfrak{m}}(R/\overline{I}) \cong H^i_{\mathfrak{m}}(R/I)(-d) \quad \text{for all } i < \dim R/I.$$

In particular, I is unmixed if and only if $\overline{I}$ is unmixed. We say that $\overline{I}$ is obtained by *basic double G-link.*

Proof. We consider the exact sequence

$$0 \longrightarrow J(-d) \stackrel{\varphi}{\longrightarrow} J \oplus I(-d) \stackrel{\psi}{\longrightarrow} \overline{I} \longrightarrow 0,$$

where the morphisms φ and ψ are defined by $\varphi(j) = (fj, j)$ and $\psi(j, i) = j - fi$. It is easy to check that this sequence is exact and the corresponding long exact sequence of local cohomology implies the claim on the cohomology groups. The second assertion now follows from the cohomological unmixedness criterion [72, Lemma 2.12]. □

Let us now justify why the name Basic Double G-link was chosen. To this end we present the above lemma in a more geometric language.

Proposition 3.2.3. *Let $S \subset \mathbb{P}^n$ be an ACM subscheme of dimension $r+1$ satisfying the property G_0. Let $X \subset S$ be an equidimensional subscheme of codimension 1 on S with no embedded components. Let F be a homogeneous element such that $I(S) : F = I(S)$. Then $I(X)$ and $I(S) + FI(X)$ are G-linked in two steps and we say that the scheme Y defined by $I(S) + FI(X)$ is a basic double G-link of X.*

Proof. See [56, Proposition 5.10]. □

Let $I \subset R$ be a homogeneous ideal of codimension c. Assume that I is standard determinantal and let $\mathcal{A}$ (resp., $\mathcal{U}$) be the $t \times (t + c - 1)$ homogeneous matrix (resp., degree matrix) associated with I. Let $\mathcal{A}'$ (resp., $\mathcal{U}'$) be the $(t-1) \times (t + c - 2)$ homogeneous matrix (resp., degree matrix) obtained by deleting the last column and the last row of $\mathcal{A}$, and denote by I' the codimension c standard determinantal ideal generated by the maximal minors of $\mathcal{A}'$. Since the multiplicity of R/I and R/I' are completely determined by the corresponding degree matrices, it is enough to consider an example of an ideal for any degree matrix. So, from now on, we take

$\mathcal{A} :=$

$$\begin{pmatrix}
x_1^{u_{1,1}} & x_2^{u_{1,2}} & \dots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \dots & 0 & 0 \\
0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \dots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \dots & 0 & 0 \\
0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \dots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \dots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} & 0 \\
0 & 0 & 0 & \dots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \dots & x_{c-1}^{u_{t,t+c-2}} & x_c^{u_{t,t+c-1}}
\end{pmatrix}$$

and $\mathcal{A}' :=$

$$\begin{pmatrix}
x_1^{u_{1,1}} & x_2^{u_{1,2}} & \dots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \dots & 0 \\
0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \dots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \dots & 0 \\
0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \dots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \dots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}}
\end{pmatrix}$$

Let $J \subset R$ be the codimension $c-1$ standard determinantal ideal generated by the maximal minors of the $t \times (t+c-2)$ homogeneous matrix

$\mathcal{B} :=$

$$\begin{pmatrix}
x_1^{u_{1,1}} & x_2^{u_{1,2}} & \dots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \dots & 0 \\
0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \dots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \dots & 0 \\
0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \dots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \dots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} \\
0 & 0 & 0 & \dots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \dots & x_{c-1}^{u_{t,t+c-2}}
\end{pmatrix}$$

obtained by deleting the last column of $\mathcal{A}$.

Analogously, let $\mathcal{A}''$ (resp., $\mathcal{U}''$) be the $(t-1)\times(t+c-2)$ homogeneous matrix (resp., degree matrix) obtained by deleting the first column and the first row of $\mathcal{A}$, and we denote by I'' the codimension c standard determinantal ideal generated by the maximal minors of $\mathcal{A}''$. Let $\mathcal{C}$ be the $t \times (t+c-2)$ homogeneous matrix obtained by deleting the first column of $\mathcal{A}$, and let $Q \subset R$ be the codimension $c-1$ standard determinantal ideal generated by the maximal minors of $\mathcal{C}$.

The ideal I is obtained from I' (resp., I'') by a basic double G-link. Indeed, we have the following lemmas.

Lemma 3.2.4. *With the above notation, we have*

(1) $I = J + x_c^{u_{t,t+c-1}} I'$ *and* $I = Q + x_1^{u_{1,1}} I''$,

(2) $e(R/I) = e(R/I') + u_{t,t+c-1} \cdot e(R/J)$ *and* $e(R/I) = e(R/I'') + u_{1,1} \cdot e(R/Q)$.

Proof. (1) The equalities of ideals are immediate.
(2) follows from [56, Lemma 4.8]. □

Lemma 3.2.5. *With the above notation, we have*

(1) $m_i = m_i(I) = m_i(I') + u_{t,t+i-1} = m'_i + u_{t,t+i-1}$ *for all* $1 \le i \le c$,
(2) $M_i = M_i(I) = M_i(I'') + u_{1,c-i+1} = M''_i + u_{1,c-i+1}$ *for all* $1 \le i \le c$,
(3) $m_i(J) = m_i(I) = m_i$ *for all* $1 \le i \le c-1$, *and*
(4) $M_i(Q) = M_i(I) = M_i$ *for all* $1 \le i \le c-1$.

Proof. This follows from Proposition 3.2.1. □

Using the fact that the ideal I is obtained from the ideal I' (resp., I'') by a basic double G-link, we can now show that Conjecture 3.0.2 is true for standard determinantal ideals of arbitrary codimension. (see [49], [67], and [68]).

Theorem 3.2.6. *Let $I \subset R$ be a standard determinantal ideal of codimension c. Then the following lower and upper bounds hold:*

(1) $e(R/I) \ge \frac{\prod_{i=1}^{c} m_i}{c!}$, *and*
(2) $e(R/I) \le \frac{\prod_{i=1}^{c} M_i}{c!}$.

Moreover, $\frac{\prod_{i=1}^{c} m_i}{c!} = e(R/I)$ *if and only if R/I has a pure resolution if and only if* $e(R/I) = \frac{\prod_{i=1}^{c} M_i}{c!}$.

Proof. As we explained above, it is enough to prove the result for the ideal I generated by the maximal minors of the $t \times (t+c-1)$ matrix

$\mathcal{A} :=$

$$\begin{pmatrix}
x_1^{u_{1,1}} & x_2^{u_{1,2}} & \dots & x_{c-1}^{u_{1,c-1}} & x_c^{u_{1,c}} & 0 & 0 & \dots & 0 & 0 \\
0 & x_1^{u_{2,2}} & x_2^{u_{2,3}} & \dots & x_{c-1}^{u_{2,c}} & x_c^{u_{2,c+1}} & 0 & \dots & 0 & 0 \\
0 & 0 & x_1^{u_{3,3}} & x_2^{u_{3,4}} & \dots & x_{c-1}^{u_{3,c+1}} & x_c^{u_{3,c+2}} & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & x_1^{u_{t-1,t-1}} & x_2^{u_{t-1,t}} & \dots & x_{c-1}^{u_{t-1,c+t-3}} & x_c^{u_{t-1,c+t-2}} & 0 \\
0 & 0 & 0 & \dots & 0 & x_1^{u_{t,t}} & x_2^{u_{t,t+1}} & \dots & x_{c-1}^{u_{t,t+c-2}} & x_c^{u_{t,t+c-1}}
\end{pmatrix}.$$

(1) We proceed by induction on the codimension c of I. If $c = 1$, then I is a principal ideal and the result is trivial. For $c = 2$, the result was proved by J. Herzog and H. Srinivasan in [46]. Assume $c \ge 3$. We will now induct on t. If $t = 1$, then I is a complete intersection ideal and hence the result is well known. Assume $t > 1$. Let $\mathcal{A}'$ (resp., $\mathcal{B}$) be the matrix obtained by deleting the last column and the last row (resp., the last column) of the matrix $\mathcal{A}$, and let I' (resp., J) be the ideal generated by the maximal minors of $\mathcal{A}'$ (resp., $\mathcal{B}$). Let m_i, m'_i, and $m_i(J)$ be the minimal shifts in the graded minimal free R-resolution of R/I, R/I', and R/J, respectively (see Proposition 3.2.1 and Lemma 3.2.5).

By Lemma 3.2.4(2),

$$e(R/I) = e(R/I') + u_{t,t+c-1} \cdot e(R/J),$$

by the induction hypothesis on c and Lemma 3.2.5(3), we have

$$e(R/J) \geq \frac{\prod_{i=1}^{c-1} m_i(J)}{(c-1)!} = \frac{\prod_{i=1}^{c-1} m_i}{(c-1)!},$$

and by the induction hypothesis on t, we have

$$e(R/I') \geq \frac{\prod_{i=1}^{c} m'_i}{(c)!}.$$

Therefore, since $m_i = m'_i + u_{t,t+i-1}$ (Lemma 3.2.5(1)), we have

$$c! \cdot e(R/I) \geq \prod_{i=1}^{c} m_i$$

if and only if

$$\begin{aligned}
\prod_{i=1}^{c} m'_i + c u_{t,t+c-1} \prod_{i=1}^{c-1} m_i &\geq \prod_{i=1}^{c} m_i = \prod_{i=1}^{c} (m'_i + u_{t,t+i-1}) \\
&= u_{t,t+c-1} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + m'_c \sum_{r=0}^{c-2} (u_{t,t+r} m_1 \dots m_r m'_{r+2} \dots m'_{c-1}) \\
&= u_{t,t+c-1} \prod_{i=1}^{c-1} m_i + \prod_{i=1}^{c} m'_i + \sum_{r=0}^{c-2} (u_{t,t+r} m_1 \dots m_r m'_{r+2} \dots m'_{c-1} m'_c)
\end{aligned}$$

if and only if

$$(c-1) u_{t,t+c-1} \prod_{i=1}^{c-1} m_i \geq \sum_{r=0}^{c-2} (u_{t,t+r} m_1 \dots m_r m'_{r+2} \dots m'_{c-1} m'_c).$$

Since, for all integers i, $1 \leq i \leq c-1$, and for all integers r, $0 \leq r \leq c-2$, we have the inequalities

$$\begin{aligned}
& m_i - m'_{i+1} \\
&\quad = (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}) \\
&\quad\quad - (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{1,i+1} + u_{2,i+2} + u_{3,i+3} + \cdots + u_{t-1,t+i-1}) \\
&\quad = (u_{2,i+1} - u_{1,i+1}) + (u_{3,i+2} - u_{2,i+2}) + \cdots + (u_{t,t+i-1} - u_{t-1,t+i-1}) \geq 0,
\end{aligned}$$

and

$$u_{t,t+c-1} \geq u_{t,t+r},$$

we obtain

$$u_{t,t+c-1} \prod_{i=1}^{c-1} m_i \geq (u_{t,t+r} m_1 \dots m_r m'_{r+2} \dots m'_{c-1} m'_c)$$

for all r, $0 \leq r \leq c-2$. So, we have

$$(c-1)u_{t,t+c-1}\prod_{i=1}^{c-1} m_i \geq \sum_{r=0}^{c-2}(u_{t,t+r}m_1 \dots m_r m'_{r+2} \dots m'_{c-1}m'_c),$$

and the lower bound follows.

(2) The upper bound is proved similarly. We again proceed by induction on the codimension c of I. If $1 \leq c \leq 2$, then the result holds. So, let us assume $c \geq 3$. We will now induct on t. If $t = 1$, then I is a complete intersection ideal and the result is true. Assume $t > 1$. Let $\mathcal{A}''$ (resp., $\mathcal{C}$) be the matrix obtained by deleting the first column and the first row (resp., the first column) of the matrix $\mathcal{A}$, and let I'' (resp., Q) be the ideal generated by the maximal minors of $\mathcal{A}''$ (resp., $\mathcal{C}$). Let M_i, M''_i, and $M_i(Q)$ be the maximal shifts in the graded minimal free R-resolution of R/I, R/I'', and R/Q, respectively.

By Lemma 3.2.4(2),

$$e(R/I) = e(R/I'') + u_{1,1} \cdot e(R/Q),$$

by the induction hypothesis on c and Lemma 3.2.5(4), we have

$$e(R/Q) \leq \frac{\prod_{i=1}^{c-1} M_i(Q)}{(c-1)!} = \frac{\prod_{i=1}^{c-1} M_i}{(c-1)!},$$

and by the induction hypothesis on t, we have

$$e(R/I'') \leq \frac{\prod_{i=1}^{c} M''_i}{(c)!}.$$

By Lemma 3.2.5(2), $M_i = M_i(I) = M_i(I'') + u_{1,c-i+1} = M''_i + u_{1,c-i+1}$ for all $1 \leq i \leq c$. Therefore, we have

$$c! \cdot e(R/I) \leq \prod_{i=1}^{c} M_i$$

if and only if

$$\begin{aligned}
&\prod_{i=1}^{c} M''_i + cu_{1,1}\prod_{i=1}^{c-1} M_i \leq \prod_{i=1}^{c} M_i = \prod_{i=1}^{c}(M''_i + u_{1,c-i+1}) \\
&= u_{1,1}\prod_{i=1}^{c-1} M_i + \prod_{i=1}^{c} M''_i + M''_c\sum_{r=0}^{c-2}(u_{1,c-r}M_1 \dots M_r M''_{r+2} \dots M''_{c-1}) \\
&= u_{1,1}\prod_{i=1}^{c-1} M_i + \prod_{i=1}^{c} M''_i + \sum_{r=0}^{c-2}(u_{1,c-r}M_1 \dots M_r M'_{r+2} \dots M''_{c-1}M''_c)
\end{aligned}$$

if and only if

$$(c-1)u_{1,1}\prod_{i=1}^{c-1} M_i \leq \sum_{r=0}^{c-2}(u_{1,c-r}M_1 \dots M_r M''_{r+2}\dots M''_{c-1}M''_c).$$

Because, for all integers i, $1 \leq i \leq c-1$, and all integers r, $0 \leq r \leq c-2$, we have

$$\begin{aligned} &M_i - M''_{i+1}\\ &\quad = (u_{1,c-i+1}+u_{2,c-i+2}+\dots+u_{t-1,t+c-i-1}+u_{t,t+c-i}+u_{t,t+c-i+1}+\dots+u_{t,t+c-1})\\ &\qquad -(u_{2,c-i+1}+u_{3,c-i+2}+\dots+u_{t,t+c-i-1}+u_{t,t+c-i}+u_{t,t+c-i+1}+\dots+u_{t,t+c-1})\\ &\quad = (u_{1,c-i+1}-u_{2,c-i+1})+(u_{2,c-i+2}-u_{3,c-i+2})+\dots+(u_{t-1,t+c-i-1}-u_{t,t+c-i-1})\\ &\quad \leq 0 \end{aligned}$$

and

$$u_{1,1} \leq u_{1,c-r},$$

we deduce

$$u_{1,1}\prod_{i=1}^{c-1} M_i \leq (u_{1,c-r}M_1 \dots M_r M''_{r+2}\dots M''_{c-1}M''_c)$$

for all r, $0 \leq r \leq c-2$. Therefore, we have

$$(c-1)u_{1,1}\prod_{i=1}^{c-1} M_i \leq \sum_{r=0}^{c-2}(u_{1,c-r}M_1 \dots M_r M''_{r+2}\dots M''_{c-1}M''_c),$$

and this completes the proof of the upper bound.

Finally, we observe that the lower inequality is an equality if and only if

$$\begin{gathered} e(R/I') = \frac{\prod_{i=1}^{c} m_i(I')}{c!}, \quad e(R/J) = \frac{\prod_{i=1}^{c-1} m_i(J)}{(c-1)!},\\ u_{r,r+i-1} - u_{r-1,r+i-1} = 0 \quad \text{for all } 2 \leq r \leq t,\\ u_{t,t+c-1} = u_{t,t+r} \quad \text{for all } 0 \leq r \leq c-2; \end{gathered}$$

and the upper inequality turns out to be an equality if and only if

$$\begin{gathered} e(R/I'') = \frac{\prod_{i=1}^{c} M_i(I'')}{c!}, \quad e(R/Q) = \frac{\prod_{i=1}^{c-1} M_i(Q)}{(c-1)!},\\ u_{r-1,r+c-i-1} - u_{r,r+c-i-1} = 0 \quad \text{for all } 2 \leq r \leq t,\\ u_{1,1} = u_{1,c-r} \quad \text{for all } 0 \leq r \leq c-2. \end{gathered}$$

Therefore, we have strict inequalities unless I has a pure resolution. □

We will illustrate by means of an example that the bounds given in Theorem 3.2.6 are optimal.

Example 3.2.7. Let $I \subset R$ be a codimension c standard determinantal ideal generated by the maximal minors of a $t \times (t+c-1)$ matrix whose all entries are homogeneous polynomials of fixed degree $1 \le d \in \mathbb{Z}$. Thus, we have

$$m_i(I) = M_i(I) = td + (i-1)d \quad \text{for all } i,\ 1 \le i \le c.$$

Therefore, we conclude that

$$\begin{aligned} e(R/I) &= \frac{\prod_{i=1}^{c} m_i(I)}{c!} = \frac{\prod_{i=1}^{c} M_i(I)}{c!} \\ &= \frac{\prod_{i=1}^{c}(td+(i-1)d)}{c!} = d^c \binom{t+c-1}{c}. \end{aligned}$$

In the last part of this chapter, we address a question which naturally arises in this context, whether, under the Cohen–Macaulay assumption, the ith total Betti number $\beta_i(R/I) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(R/I)$ can be bounded above by a function of the maximal shifts in the minimal graded free R-resolution of R/I as well as bounded below by a function of the minimal shifts. In [78], T. Römer made a natural guess,

$$\prod_{1\le j<i} \frac{m_j}{m_i - m_j} \prod_{i<j\le c} \frac{m_j}{m_j - m_i} \le \beta_i(R/I) \le \prod_{1\le j<i} \frac{M_j}{M_i - M_j} \prod_{i<j\le c} \frac{M_j}{M_j - M_i} \tag{3.4}$$

for $i = 1, \dots, c$, and he showed that these bounds hold if I is a complete intersection or componentwise linear. Moreover, in these cases we have equality above or below if and only if R/I has a pure resolution. Unfortunately, these bounds are not always valid (see [78, Example 3.1]). For Cohen–Macaulay algebras with strictly quasi-pure resolution (i.e., $m_i > M_{i-1}$ for all i), he showed that

$$\prod_{1\le j<i} \frac{m_j}{M_i - m_j} \prod_{i<j\le c} \frac{m_j}{M_j - m_i} \le \beta_i(R/I) \le \prod_{1\le j<i} \frac{M_j}{m_i - M_j} \prod_{i<j\le c} \frac{M_j}{m_j - M_i} \tag{3.5}$$

for $i = 1, \dots, c$, and again we have equalities if and only if R/I has a pure resolution. Notice that

$$\prod_{1\le j<i} \frac{M_j}{m_i - M_j} \prod_{i<j\le c} \frac{M_j}{m_j - M_i}$$

may be negative and thus, in general,

$$\prod_{1\le j<i} \frac{M_j}{m_i - M_j} \prod_{i<j\le c} \frac{M_j}{m_j - M_i}$$

is not a good candidate for being an upper bound. In [78], T. Römer suggested as upper bound

$$\beta_i(R/I) \le \frac{1}{(i-1)! \cdot (c-i)!} \prod_{j\ne i} M_j \tag{3.6}$$

for $i = 1, \dots, c$, and he proved that the lower bound in (3.5) and the upper bound in (3.6) hold if R/I is Cohen–Macaulay of codimension 2 and Gorenstein of codimension 3. We will now prove that the lower bound in (3.5) and the upper bound in (3.6) work for standard determinantal ideals of arbitrary codimension c.

Remark 3.2.8. Let $I \subset R$ be a standard determinantal ideal. It is worthwhile to point out that the ith total Betti number $\beta_i(R/I)$ in the minimal free R-resolution of R/I depend only upon the size $t \times (t+c-1)$ of the homogeneous matrix $\mathcal{A}$ associated with I (see Proposition 3.2.1).

Theorem 3.2.9. *Let $I \subset R$ be a standard determinantal ideal of codimension c. Then, we have*

$$\prod_{1 \le j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \le c} \frac{m_j}{M_j - m_i} \le \beta_i(R/I) \le \frac{1}{(i-1)! \cdot (c-i)!} \prod_{j \neq i} M_j \tag{3.7}$$

for $i = 1, \dots, c$. In addition, the bounds are reached for all i if and only if R/I has a pure resolution if and only if $u_{i,j} = u_{r,s}$ for all $1 \le i, r \le t$ and $1 \le j, s \le t+c-1$.

Proof. We will first prove the result for J, where $J \subset R$ is a standard determinantal ideal of codimension c whose associated matrix $\mathcal{A}$ is a $t \times (t+c-1)$ matrix with all its entries linear forms. In this case, for all $i = 1, \dots, c$, we have (see Proposition 3.2.1),

$$m_i(J) = M_i(J) = t+i-1 \quad \text{and} \quad \beta_i(R/J) = \binom{t+c-1}{t+i-1}\binom{t+i-2}{i-1}.$$

Therefore, R/J has a pure resolution and it follows from [45] and [51] that

$$\begin{aligned}
&\prod_{1 \le j < i} \frac{m_j(J)}{M_i(J) - m_j(J)} \prod_{i < j \le c} \frac{m_j(J)}{M_j(J) - m_i(J)} \\
&\quad = \prod_{1 \le j < i} \frac{t+j-1}{i-j} \prod_{i < j \le c} \frac{t+j-1}{j-i} \\
&\quad = \beta_i(R/J) \\
&\quad = \prod_{1 \le j < i} \frac{t+j-1}{i-j} \prod_{i < j \le c} \frac{t+j-1}{j-i} \\
&\quad = \prod_{1 \le j < i} \frac{M_j(J)}{m_i(J) - M_j(J)} \prod_{i < j \le c} \frac{M_j(J)}{m_j(J) - M_i(J)} \\
&\quad = \frac{1}{(i-1)! \cdot (c-i)!} \prod_{j \neq i} M_j(J).
\end{aligned}$$

We will now prove the general case. Let I be a standard determinantal ideal of codimension c with associated degree matrix $\mathcal{U} = (u_{i,j})_{i=1,\dots,t}^{j=1,\dots,t+c-1}$. Since, for all $i = 1, \ldots, c$, we have

$$M_i(I) \geq m_i(I) \geq t + i - 1 = m_i(J) = M_i(J),$$

it follows from Proposition 3.2.1(3) and Remark 3.2.8 that

$$\begin{aligned}\beta_i(R/I) = \beta_i(R/J) &= \prod_{1\leq j<i} \frac{t+j-1}{i-j} \prod_{i<j\leq c} \frac{t+j-1}{j-i} \\ &= \frac{1}{(i-1)!\cdot(c-i)!} \prod_{j\neq i} M_j(J) \leq \frac{1}{(i-1)!\cdot(c-i)!} \prod_{j\neq i} M_j(I),\end{aligned}$$

and this completes the proof of the upper bound.

Let us now prove the lower bound. Recall that if $r \geq m$ and $s \geq n$, then $u_{r,s} \geq u_{m,n}$. Therefore, for $1 \leq j < i$, we get, using Proposition 3.2.1, that

$$\begin{aligned} m_j &= u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1} \\ &\leq (t+j-1)u_{t,t+j-1}, \end{aligned}$$

$$\begin{aligned} M_i - m_j &= u_{1,c-i+1} + u_{2,c-i+2} + \cdots + u_{t,t+c-i} + u_{t,t+c-i+1} + u_{t,t+c-i+2} + \cdots \\ &\quad + u_{t,t+c-1} - (u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1}) \\ &\geq u_{t,t+c-i+j} + u_{t,t+c-i+j+1} + \cdots + u_{t,t+c-1} \\ &\geq (i-j)u_{t,t+c-i+j}, \end{aligned}$$

and

$$\frac{m_j}{M_i - m_j} \leq \frac{(t+j-1)u_{t,t+j-1}}{(i-j)u_{t,t+c-i+j}} \leq \frac{t+j-1}{i-j}.$$

For $i < j \leq c$, we have

$$\begin{aligned} m_j &= u_{1,1} + u_{1,2} + \cdots + u_{1,j} + u_{2,j+1} + u_{3,j+2} + \cdots + u_{t,t+j-1} \\ &= \begin{cases} u_{1,1} + u_{1,2} + \cdots + u_{1,t+i-1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,t+j-1}, \\ \hfill \text{if } t+i \leq j, \\ \\ u_{1,1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t+i-j,t+i-1} + \cdots + u_{t,t+j-1}, \\ \hfill \text{if } t+i > j; \end{cases} \end{aligned}$$

$$\begin{aligned} M_j - m_i &= u_{1,c-j+1} + u_{2,c-j+2} + \cdots + u_{t,t+c-j} + u_{t,t+c-j+1} \\ &\quad + u_{t,t+c-j+2} + \cdots + u_{t,t+c-1} \\ &\quad - (u_{1,1} + u_{1,2} + \cdots + u_{1,i} + u_{2,i+1} + u_{3,i+2} + \cdots + u_{t,t+i-1}) \\ &\geq u_{t,t+c+i-j} + u_{t,t+c+i-j+1} + \cdots + u_{t,t+c-1} \\ &\geq \begin{cases} u_{1,t+i} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,t+j-1}, & \text{if } t+i \leq j, \\ u_{t+i-j+1,t+i} + \cdots + u_{t,t+j-1}, & \text{if } t+i > j. \end{cases} \end{aligned}$$

Therefore, if $i < j \leq c$ and $t + i \leq j$, we get

$$\begin{aligned}\frac{m_j}{M_j - m_i} &\leq \frac{u_{1,1} + u_{1,2} + \cdots + u_{1,t+i-1}}{u_{1,t+i} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t,t+j-1}} + 1 \\ &\leq \frac{(t+i-1)u_{1,t+i-1}}{(j-i)u_{1,t+i}} + 1 \\ &\leq \frac{t+i-1}{j-i} + 1 \\ &= \frac{t+j-1}{j-i},\end{aligned}$$

and if $i < j \leq c$ and $t + i > j$, we get

$$\begin{aligned}\frac{m_j}{M_j - m_i} &\leq \frac{u_{1,1} + \cdots + u_{1,j} + u_{2,j+1} + \cdots + u_{t+i-j,t+i-1}}{u_{t+i-j+1,t+i} + \cdots + u_{t,t+j-1}} + 1 \\ &\leq \frac{(t+i-1)u_{t+i-j,t+i-1}}{(j-i)u_{t+i-j+1,t+i}} + 1 \\ &\leq \frac{t+i-1}{j-i} + 1 \\ &= \frac{t+j-1}{j-i}.\end{aligned}$$

Hence, putting altogether, we obtain

$$\begin{aligned}\prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} &\leq \prod_{1 \leq j < i} \frac{t+j-1}{i-j} \prod_{i < j \leq c} \frac{t+j-1}{j-i} \\ &= \beta_i(R/J) \\ &= \beta_i(R/I),\end{aligned}$$

and this completes the proof of the lower bound. Checking the inequalities we easily see that we have equality above and below for all $1 \leq i \leq c$ if and only if R/I has a pure resolution. This concludes the proof of the theorem. ☐

Remark 3.2.10. Since a complete intersection ideal I of arbitrary codimension and Cohen–Macaulay ideals of codimension 2 are examples of standard determinantal ideals, we recover [78, Theorem 2.1 and Corollary 4.2.].

This page intentionally left blank

Chapter 4

Unobstructedness and Dimension of Families of Standard Determinantal Ideals

Let $\mathbb{P}^r$ be the r-dimensional projective space over a field K and fix a numerical polynomial $p(t) = \sum_{i=0}^{r} a_i \binom{t+r}{i} \in \mathbb{Q}[t]$ with $a_i \in \mathbb{Z}$ for all i. We consider the contravariant functor

$$\underline{\mathrm{Hilb}}^r_{p(t)} : (Sch/K) \longrightarrow (Sets)$$

defined by

$$\underline{\mathrm{Hilb}}^r_{p(t)}(S) := \{ \text{ flat families } \mathcal{X} \subset \mathbb{P}^r \times S \text{ of closed subschemes of } \mathbb{P}^r \text{ parameterized by } S \text{ with fibers having Hilbert polynomial } p(t) \}.$$

Such functor is called the *Hilbert functor*. In 1960, A. Grothendieck proved (see [35, Théorème 3.2]) the following.

There exists a unique scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^r)$ called the *Hilbert scheme* which parameterizes a flat family

$$\pi : \mathcal{W} \subset \mathbb{P}^r \times \mathrm{Hilb}^{p(t)}(\mathbb{P}^r) \longrightarrow \mathrm{Hilb}^{p(t)}(\mathbb{P}^r)$$

of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, and has the following universal property: For every flat family $f : \mathcal{X} \subset \mathbb{P}^r_S = \mathbb{P}^r \times S \longrightarrow S$ of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, there is a unique morphism $g : S \longrightarrow \mathrm{Hilb}^{p(t)}(\mathbb{P}^r)$, called the classification map for the family f such that π induces f by base change; i.e., $\mathcal{X} = S \times_{\mathrm{Hilb}^{p(t)}(\mathbb{P}^r)} \mathcal{W}$.

In the usual language of categories we say that the pair $(\mathrm{Hilb}^{p(t)}(\mathbb{P}^r), \pi)$ represents the moduli functor $\underline{\mathrm{Hilb}}^r_{p(t)}$, and π is the universal family. Once the existence of the Hilbert scheme $\operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$ is established, we would like to know what does the Hilbert scheme look like? Is it smooth or irreducible? What is its dimension? etc.

It is a classical result of R. Hartshorne that $\operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$ is connected (see [38]). It is also well known that if we denote by $[X]$ the K-point which parameterizes a closed subscheme $X \subset \mathbb{P}^r$ and we denote by $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^r}$ the ideal sheaf of X in $\mathbb{P}^r$, then there is a canonical isomorphism of K-vector spaces

$$T_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^r) \cong H^0(X, \mathcal{N}_X), \tag{4.1}$$

where

$$\mathcal{N}_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$$

is the normal sheaf of X in $\mathbb{P}^r$ and $T_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$ is the Zariski tangent space of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$ at $[X]$; and the obstruction space of $\mathcal{O}_{\operatorname{Hilb}^{p(t)}(\mathbb{P}^r),[X]}$ is a subspace of $H^1(X, \mathcal{N}_X)$. In spite of the great progress made during the last decades in the problem of studying the local and global structure of the Hilbert scheme $\operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$, there is no general answer about the smoothness, dimension, and number of irreducible components of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^r)$.

In this chapter we denote by $W(\underline{b};\underline{a}) \subset \operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ (resp., $W_s(\underline{b};\underline{a})$) the locus of good (resp., standard) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c defined by the maximal minors of a $t \times (t+c-1)$ matrix

$$\mathcal{A} = (f_{ij})_{j=0,\dots,t+c-2}^{i=1,\dots,t}$$

where $f_{ij} \in K[x_0, x_1, \dots, x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$, and we address the following three fundamental problems:

(1) to determine the dimension of $W(\underline{b};\underline{a})$ (resp., $W_s(\underline{b};\underline{a})$) in terms of a_j and b_i,

(2) whether the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$, and

(3) when $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is generically smooth along $W(\underline{b};\underline{a})$.

Concerning problem (1) we give an upper bound for the dimension of $W(\underline{b};\underline{a})$ (resp., $W_s(\underline{b};\underline{a})$) which works for all integers $a_0, a_1, \dots, a_{t+c-2}$ and $b_1, \dots, b_t$, and we conjecture that this bound is sharp. The conjecture is proved for $2 \le c \le 5$, and for $c \ge 6$ under some numerical restriction on $a_0, a_1, \dots, a_{t+c-2}$ and $b_1, \dots, b_t$. For problems (2) and (3) we have an affirmative answer for $2 \le c \le 4$ and $n \ge 2$, and for $c \ge 5$ under certain numerical assumptions.

4.1 Families of codimension 2 Cohen–Macaulay algebras

Definition 4.1.1. A closed subscheme $X \subset \mathbb{P}^n$ is called *unobstructed* if $\operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ is smooth at the point $[X]$; i.e.,

$$h^0(X, \mathcal{N}_X) = \dim T_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^n) \\ = \dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^n).$$

For a positive-dimensional ACM scheme $X \subset \mathbb{P}^n$ the vanishing of the algebra cohomology ${}_0H^2(R, R/I(X), R/I(X))$ is sufficient but not necessary for the unobstructedness.

The first important contribution to the problem of determining the unobstructedness and the dimension of the family of standard (resp., good) determinantal schemes is due to G. Ellinsgrud [25]. In 1975, he proved that every ACM, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is unobstructed (i.e., the corresponding point in the Hilbert scheme $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+2})$ is smooth) provided $n \geq 1$, and he also computed the dimension of the Hilbert scheme at $[X]$. Recall also that the homogeneous ideal of an ACM, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is given by the maximal minors of a $t \times (t+1)$ homogeneous matrix, the Hilbert–Burch matrix. That is, such a scheme is standard determinantal. We will devote the first section of this chapter to Ellingsrud's theorem.

Theorem 4.1.2 (Ellingsrud's theorem). *Let $X_0 \subset Y_0 = \mathbb{P}^n_K$ be an ACM closed subscheme of codimension 2, and assume $\dim X_0 \geq 1$. Suppose we are given a local Artinian ring C', an ideal $J \subset C'$, and its quotient $C = C'/J$. Suppose we are given a closed subscheme $X \subset Y = \mathbb{P}^n_C$, flat over C and with $X \times_C K = X_0$. Then*

(1) *there is an $r \times (r+1)$ matrix φ of homogeneous elements of $S = C[x_0, \dots, x_n]$ whose maximal minors f_i generate the ideal I of X, and give a resolution*

$$0 \longrightarrow \oplus_{i=1}^r S(-b_i) \xrightarrow{\varphi} \oplus_{j=1}^{r+1} S(-a_j) \xrightarrow{f} S \longrightarrow S/I \longrightarrow 0,$$

where $f = (f_i, \dots, f_{r+i})$;

(2) *for any lifting φ' of φ to $S' = C'[x_0, \dots, x_n]$, taking $f' = (f'_i, \dots, f'_{r+i}$ to be the $r \times r$ minors of φ', and I' the ideal generated by $f'_i, \dots, f'_{r+i}$, we have an exact sequence*

$$0 \longrightarrow \oplus_{i=1}^r S'(-b_i) \xrightarrow{\varphi'} \oplus_{j=1}^{r+1} S'(-a_j) \xrightarrow{f'} S' \longrightarrow S'/I' \longrightarrow 0$$

and defines a closed subscheme $X' \subset Y' = \mathbb{P}^n_{C'}$, with $X' \times_{C'} C = X$; and

(3) *any lifting X' of X to Y', flat over C', with $X' \times_{C'} C = X$, arises by lifting φ as in* (2).

Proof. See [25, Théorème 2] or [40, Theorem 8.10.]. □

Corollary 4.1.3. *The Hilbert scheme* $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$ *is smooth at a point corresponding to a codimension* 2 *ACM closed subscheme* $X \subset \mathbb{P}^n$. *Moreover, if*

$$0 \longrightarrow \oplus_{i=1}^{r} R(-b_i) \longrightarrow \oplus_{j=1}^{r+1} R(-a_j) \longrightarrow R \longrightarrow R/I(X) \longrightarrow 0$$

is a free R*-resolution of the homogeneous ideal of* X, *we have*

$$\begin{aligned}\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n) = &\sum_{a_j \geq b_i} \binom{a_j - b_i + n}{n} + \sum_{b_i \geq a_j} \binom{b_i - a_j + n}{n} \\ &- \sum_{a_j \geq a_i} \binom{a_j - a_i + n}{n} - \sum_{b_i \geq b_j} \binom{b_i - b_j + n}{n} + 1.\end{aligned}$$

Proof. If $n \geq 3$, the smoothness of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$ at $[X]$ follows from Theorem 4.1.2 and the infinitesimal lifting property of smoothness. Let us compute the dimension. Since $\mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$ is smooth at $[X]$, we have

$$\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n) = \dim T_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n).$$

Using (4.1) we get

$$\begin{aligned}\dim T_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n) &= H^0(X, \mathcal{N}_X) \\ &= \dim \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X),\end{aligned}$$

where $\mathcal{I}_X$ is the ideal sheaf of X. Applying $\mathrm{Hom}(\cdot, \mathcal{I}_X)$ to the exact sequence

$$0 \longrightarrow \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(-b_i) \longrightarrow \oplus_{j=1}^{r+1} \mathcal{O}_{\mathbb{P}^n}(-a_j) \longrightarrow \mathcal{I}_X \longrightarrow 0,$$

and taking into account that $\dim \mathrm{Hom}(\mathcal{I}_X, \mathcal{I}_X) = 1$, we obtain

$$\begin{aligned}\dim \mathrm{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) &= \sum_{j=1}^{r+1} h^0(\mathcal{I}_X(a_j)) - \sum_{i=1}^{r} h^0(\mathcal{I}_X(b_i)) + 1 \\ &= \sum_{a_j \geq b_i} \binom{a_j - b_i + n}{n} + \sum_{b_i \geq a_j} \binom{b_i - a_j + n}{n} \\ &\quad - \sum_{a_j \geq a_i} \binom{a_j - a_i + n}{n} - \sum_{b_i \geq b_j} \binom{b_i - b_j + n}{n} + 1.\end{aligned}$$

If $n = 2$, then X is a 0-dimensional scheme contained in an affine space $\mathbb{A}^2$ and the smoothness was proved by J. Fogarty in [26]. □

Example 4.1.4. Let $H(3,0) \subset \mathrm{Hilb}^{3t-1}(\mathbb{P}^3)$ be the open subset parameterizing equidimensional curves of degree 3 and arithmetic genus 0 without embedded components. Any twisted cubic $[C] \in H(3,0)$ is ACM and its homogeneous ideal $I(C)$ has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R/I(C) \longrightarrow 0.$$

Therefore, $H(3,0) = W(0,0;1,1,1)$, and we conclude that $H(3,0)$ is smooth connected of dimension 12.

4.2 Unobstructedness and dimension of families of determinantal schemes

In this section, using induction on the codimension, we extend Ellingsrud's theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary codimension. We will first write down an upper bound for the dimension of the locus $W(\underline{b};\underline{a})$ (resp., $W_s(\underline{b};\underline{a})$) of good (resp., standard) determinantal subschemes $X \subset \mathbb{P}^{n+c}$ of codimension c inside the Hilbert scheme $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$, where $p(s) \in \mathbb{Q}[s]$ is the Hilbert polynomial of X which can be computed explicitly using the minimal free R-resolution of $R/I(X)$ given by the Eagon–Northcott complex (see Proposition 1.2.16(2) and [11, Proposition 2.4]). Then, we will analyze when the mentioned upper bound is sharp and we will discuss under which conditions the closure of $W(\underline{b};\underline{a})$ in $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$ is a generically smooth, irreducible component of $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$.

Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of codimension $c \geq 2$ defined by the vanishing of the maximal minors of a $t \times (t+c-1)$ matrix

$$\mathcal{A} = (f_{ji})_{i=1,\ldots,t}^{j=0,\ldots,t+c-2},$$

where $f_{ji} \in K[x_0, \ldots, x_{n+c}]$ are homogeneous polynomials of degree $a_j - b_i$, and let $A = R/I(X)$ be the homogeneous coordinate ring of X. The matrix $\mathcal{A}$ defines a morphism of locally free sheaves,

$$\varphi : \mathcal{F} := \bigoplus_{i=1}^{t} \mathcal{O}_{\mathbb{P}^{n+c}}(b_i) \longrightarrow \mathcal{G} := \bigoplus_{j=0}^{t+c-2} \mathcal{O}_{\mathbb{P}^{n+c}}(a_j),$$

and we may assume, without loss of generality, that φ is minimal; i.e., $f_{ji} = 0$ for all i, j with $b_i = a_j$.

Now, we consider the affine scheme $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{F}, \mathcal{G})$ whose rational points are the morphisms from $\mathcal{F}$ to $\mathcal{G}$. Let $\mathbb{Y}$ be the nonempty, open, irreducible subscheme of $\mathbb{V}$ whose rational points are the morphisms $\varphi_\lambda : \mathcal{F} \longrightarrow \mathcal{G}$ such that their associated homogeneous matrix $\mathcal{A}_\lambda$ defines a good determinantal subscheme $X_\lambda \subset \mathbb{P}^{n+c}$. The Eagon–Northcott complex of the universal morphism

$$\Psi : pr_2^* \mathcal{F} \longrightarrow pr_2^* \mathcal{G}$$

on $\mathbb{Y} \times \mathbb{P}^{n+c}$ (where $pr_2 : \mathbb{Y} \times \mathbb{P}^{n+c} \longrightarrow \mathbb{P}^{n+c}$ is the natural projection) induces a morphism,

$$f : \mathbb{Y} \longrightarrow W(\underline{b};\underline{a})$$

which is defined by $f(\varphi_\lambda) := X_\lambda$ on closed points. We consider the affine group scheme $G := \mathbb{A}\mathrm{ut}(\mathcal{F}) \times \mathbb{A}\mathrm{ut}(\mathcal{G})$ which is an irreducible open dense subset of

$$G \subset \mathrm{Hom}(\mathcal{F}, \mathcal{F}) \times \mathrm{Hom}(\mathcal{G}, \mathcal{G}) \cong K^{\Upsilon},$$

where $\Upsilon = \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{i,j} \binom{a_i - a_j + n + c}{n+c}$. The affine group scheme G operates on $\mathbb{Y}$,

$$\sigma : G \times \mathbb{Y} \longrightarrow \mathbb{Y}; \quad ((\alpha, \beta), \varphi_\lambda) \mapsto \beta\varphi_\lambda\alpha^{-1}.$$

The action σ is compatible with the morphism f. Thus, at least set-theoretically $f : \mathbb{Y} \longrightarrow W(\underline{b}; \underline{a})$ induces a surjective map from the orbit set $\mathbb{Y}//G$ to $W(\underline{b}; \underline{a})$. Moreover, since the map from $\mathbb{Y}$ to the closure $\overline{W(\underline{b}; \underline{a})}$ in $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$ is dominant, we get that $W(\underline{b}; \underline{a})$ is irreducible and we have (small letters denote dimension)

$$\dim W(\underline{b}; \underline{a}) \leq \hom_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{F}, \mathcal{G}) - \operatorname{aut}(\mathcal{G}) - \operatorname{aut}(\mathcal{F}) + \dim(G_\lambda), \tag{4.2}$$

where

$$G_\lambda = \{(\delta, \tau) \in \mathbb{A}\mathrm{ut}(\mathcal{F}) \times \mathbb{A}\mathrm{ut}(\mathcal{G}) \mid \tau\varphi_\lambda\delta^{-1} = \varphi_\lambda\}$$

is the isotropy group of any closed point $\varphi_\lambda \in \mathbb{Y}$. The following proposition holds.

Proposition 4.2.1. *For all $\varphi_\lambda \in \mathbb{Y}$, we have (we let $\binom{n+a}{a} = 0$ for $a < 0$, as usual)*

$$\dim(G_\lambda) = \operatorname{aut}(\mathcal{B}_\lambda) + \sum_{j,i} \binom{b_i - a_j + n + c}{n + c}, \tag{4.3}$$

where $\mathcal{B}_\lambda = \operatorname{coker}(\varphi_\lambda)$.

Proof. Since any automorphism $\beta \in \mathbb{A}\mathrm{ut}(\mathcal{B}_\lambda)$ lifts and determines automorphisms $\beta_\mathcal{G} \in \mathbb{A}\mathrm{ut}(\mathcal{G})$ and $\beta_\mathcal{F} \in \mathbb{A}\mathrm{ut}(\mathcal{F})$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi_\lambda} & \mathcal{G} & \longrightarrow & \mathcal{B}_\lambda & \longrightarrow & 0 \\ & & \beta_\mathcal{F} \downarrow & & \downarrow \beta_\mathcal{G} & & \downarrow \beta & & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi_\lambda} & \mathcal{G} & \longrightarrow & \mathcal{B}_\lambda & \longrightarrow & 0. \end{array}$$

Now we easily can see that G_λ contains a nonempty open subset of $\mathbb{A}\mathrm{ut}(\mathcal{B}_\lambda) \times \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{G}, \mathcal{F})$.

Indeed, for a general $\beta \in \mathbb{A}\mathrm{ut}(\mathcal{B}_\lambda)$ and a general $\rho \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{G}, \mathcal{F})$, the pair $(\delta, \tau) = (\beta_\mathcal{F} + \rho\varphi_\lambda, \beta_\mathcal{G} + \varphi_\lambda\rho)$ belongs to the isotropy group of φ because

$$\begin{aligned} (\beta_\mathcal{G} + \varphi_\lambda\rho)\varphi_\lambda(\beta_\mathcal{F} + \rho\varphi_\lambda)^{-1} &= (\beta_\mathcal{G}\varphi_\lambda + \varphi_\lambda\rho\varphi_\lambda)(\beta_\mathcal{F} + \rho\varphi_\lambda)^{-1} \\ &= (\varphi_\lambda\beta_\mathcal{F} + \varphi_\lambda\rho\varphi_\lambda)(\beta_\mathcal{F} + \rho\varphi_\lambda)^{-1} \\ &= \varphi_\lambda(\beta_\mathcal{F} + \rho\varphi_\lambda)(\beta_\mathcal{F} + \rho\varphi_\lambda)^{-1} \\ &= \varphi_\lambda. \end{aligned}$$

Conversely, we claim that if $(\delta, \tau) \in G_\lambda$, then there exists $\beta \in \mathbb{A}\mathrm{ut}(\mathcal{B}_\lambda)$ and $\rho \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{G}, \mathcal{F})$ such that $(\delta, \tau) = (\beta_\mathcal{F} + \rho\varphi_\lambda, \beta_\mathcal{G} + \varphi_\lambda\rho)$. To this end, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi_\lambda} & \mathcal{G} & \longrightarrow & \mathcal{B}_\lambda & \longrightarrow & 0 \\ & & \delta \downarrow & & \downarrow \tau & & \downarrow \beta & & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi_\lambda} & \mathcal{G} & \longrightarrow & \mathcal{B}_\lambda & \longrightarrow & 0, \end{array}$$

where we denote by $\beta \in \mathbb{A}\mathrm{ut}(\mathcal{B}_\lambda)$ the induced automorphism. Then $Im(\tau - \beta_{\mathcal{G}}) \subset Im(\varphi_\lambda)$ and thus there exists $\rho \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{G}, \mathcal{F})$ such that $\tau = \beta_{\mathcal{G}} + \varphi_\lambda \rho$. It follows that $\varphi_\lambda \delta = \tau \varphi_\lambda = (\beta_{\mathcal{G}} + \varphi_\lambda \rho)\varphi_\lambda = \varphi_\lambda(\beta_{\mathcal{F}} + \rho\varphi_\lambda)$, i.e., $\delta = \beta_{\mathcal{F}} + \rho\varphi_\lambda$. Hence, we have

$$\begin{aligned}\dim(G_\lambda) &= \mathrm{aut}(\mathcal{B}_\lambda) + \mathrm{hom}_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{G}, \mathcal{F}) \\ &= \mathrm{aut}(\mathcal{B}_\lambda) + \sum_{i,j} h^0(\mathcal{O}_{\mathbb{P}^{n+c}}(b_j - a_i)) \\ &= \mathrm{aut}(\mathcal{B}_\lambda) + \sum_{j,i} \binom{b_j - a_i + n + c}{n + c}. \end{aligned} \qquad \square$$

Therefore, using inequality (4.2) and the above proposition, we obtain

$$\begin{aligned}\dim W(\underline{b}; \underline{a}) \le &\sum_{i,j} \binom{a_i - b_j + n + c}{n + c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n + c} \\ &- \sum_{i,j} \binom{b_i - b_j + n + c}{n + c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n + c} + \mathrm{aut}(\mathcal{B}_\lambda). \end{aligned} \tag{4.4}$$

We are led to pose the following question.

Question 4.2.2. *Does G act transitively on the fibers of $f : \mathbb{Y} \longrightarrow W(\underline{b}; \underline{a})$?*

Note that if G acts transitively on the fibers of f, then the surjective map from the orbit set $\mathbb{Y}//G$ to $W(\underline{b}; \underline{a})$ will be an isomorphism.

Our next goal is to bound the dimension $\mathrm{aut}(\mathcal{B})$ in terms of a_j and b_i, where $\mathcal{B} = \mathrm{coker}(\varphi)$ and φ is a closed point of $\mathbb{Y}$ (see Proposition 4.2.5). To this end we need to fix some more notation.

Let $\mathcal{A}_i$ be the matrix obtained by deleting the last $c - i$ columns of $\mathcal{A}$. The matrix $\mathcal{A}_i$ defines a morphism,

$$\varphi_i : F = \bigoplus_{s=1}^{t} R(b_s) \longrightarrow G_i := \bigoplus_{j=0}^{t+i-2} R(a_j) \tag{4.5}$$

of free R-modules, and let B_i be the cokernel of φ_i. Put $\varphi = \varphi_0$, $G = G_c$, and $B = B_c$. Let M_i be the cokernel of $\varphi_i^* = \mathrm{Hom}_R(\varphi_i, R)$, i.e., let the sequence

$$G_i^* \xrightarrow{\varphi_i^*} F^* \longrightarrow M_i \cong \mathrm{Ext}^1_R(B_i, R) \longrightarrow 0 \tag{4.6}$$

be exact. If $D_i \cong R/I_{D_i}$ is the K-algebra given by the maximal minors of $\mathcal{A}_i$ and $X_i = \mathrm{Proj}(D_i)$ (i.e., we have the flag $R \twoheadrightarrow D_2 \twoheadrightarrow D_3 \twoheadrightarrow \cdots \twoheadrightarrow D_c = A$), then M_i is a D_i-module and there is an exact sequence

$$0 \longrightarrow D_i \longrightarrow M_i(a_{t+i-1}) \longrightarrow M_{i+1}(a_{t+i-1}) \longrightarrow 0 \tag{4.7}$$

in which $D_i \longrightarrow M_i(a_{t+i-1})$ is the regular section which defines D_{i+1} (see [61]). Indeed,

$$0 \longrightarrow M_i(a_{t+i-1})^* = \operatorname{Hom}_{D_i}(M_i(a_{t+i-1}), D_i) \longrightarrow D_i \longrightarrow D_{i+1} \longrightarrow 0 \qquad (4.8)$$

and we may put $I_i := I_{D_{i+1}/D_i} = M_i(a_{t+i-1})^*$. A free R-resolution of M_i is given by Proposition 1.2.16, and we get, in particular, that M_i is a maximal Cohen–Macaulay D_i-module. Using the exact sequence (4.8), we see that I_i is also a maximal Cohen–Macaulay D_i-module. Proposition 1.2.16(3) also gives us $K_{D_i}(n+c+1) \cong S_{i-1}M_i(\ell_i)$ where $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{q=1}^{t} b_q$.

In what follows we always let $Z_i \subset X_i$ be some closed subset such that $U_i = X_i - Z_i \hookrightarrow \mathbb{P}^{n+c}$ is a local complete intersection. By the well-known fact that the fitting ideal of M_i is equal to $I_{t-1}(\varphi_i)$, we get that $\tilde{M}_i$ is locally free of rank 1 precisely on $X_i - V(I_{t-1}(\varphi_i))$ (see [9, Lemma 1.4.8]). Since the set of nonlocally complete intersection points of $X_i \hookrightarrow \mathbb{P}^{n+c}$ is precisely $V(I_{t-1}(\varphi_i))$ by, e.g., [86, Lemma 1.8], we get that $U_i \subset X_i - V(I_{t-1}(\varphi_i))$ and that $\tilde{M}_i$ and $\mathcal{I}_{X_i}/\mathcal{I}_{X_i}^2$ are locally free on U_i.

Finally note that there is a close relation between $M_{i+1}(a_{t+i-1})$ and the normal module $N_{D_{i+1}/D_i} := \operatorname{Hom}_{D_i}(I_i, D_{i+1})$ of the quotient $D_i \to D_{i+1}$. If we suppose $\operatorname{depth}_{I(Z_i)} D_i \geq 2$, we get, by applying $\operatorname{Hom}_{D_i}(I_i, .)$ to the exact sequence (4.8), that

$$0 \longrightarrow D_i \longrightarrow M_i(a_{t+i-1}) \longrightarrow N_{D_{i+1}/D_i} \qquad (4.9)$$

is exact. Hence, we have an injection $M_{i+1}(a_{t+i-1}) \hookrightarrow N_{D_{i+1}/D_i}$, which in the case that $\operatorname{depth}_{I(Z_i)} D_i \geq 3$ leads to an isomorphism $M_{i+1}(a_{t+i-1}) \cong N_{D_{i+1}/D_i}$. Indeed, this follows from the more general fact (by letting $M = N = I_i$) that if M and N are finitely generated D-modules such that $\operatorname{depth}_{I(Z)} M \geq r+1$ and $\tilde{N}$ is locally free on $U := X - Z$ ($X = \operatorname{Proj}(D)$), then the natural map

$$\operatorname{Ext}^i_D(N, M) \longrightarrow H^i_*(U, \mathcal{H}om_{\mathcal{O}_X}(\tilde{N}, \tilde{M})) \cong H^{i+1}_{I(Z)}(\operatorname{Hom}_D(N, M)) \qquad (4.10)$$

is an isomorphism, (resp., an injection) for $i < r$ (resp., $i = r$), cf. [36, Exp. VI].

Lemma 4.2.3. *Let M be an R-module. With the above notation, the sequence*

$$0 \to \operatorname{Hom}_R(M_i, M) \to F \otimes_R M \to G_i \otimes_R M \to B_i \otimes_R M \to 0$$

is exact and $\operatorname{Hom}_R(M_i, M) = \operatorname{Tor}_1^R(B_i, M)$.

Proof. We apply $\operatorname{Hom}_R(., R)$ to the exact sequence

$$0 \longrightarrow F \longrightarrow G_i \longrightarrow B_i \longrightarrow 0, \qquad (4.11)$$

and we get

$$0 \to \operatorname{Hom}_R(B_i, R) \to G_i^* \to F^* \to \operatorname{Ext}^1_R(B_i, R) = M_i \to 0.$$

Hence,

$$0 \to \mathrm{Hom}_R(M_i, M) \to \mathrm{Hom}_R(F^*, M) \cong F \otimes M \to \mathrm{Hom}_R(G_i^*, M) \cong G_i \otimes M,$$

and we get the first exact sequence and $\mathrm{Hom}_R(M_i, M) = \mathrm{Tor}_1^R(B_i, M)$ by applying $(.) \otimes_R M$ to (4.11). □

Lemma 4.2.4. *With the notations above, if $X \subset \mathbb{P}^{n+c}$ is a good determinantal scheme, then $\mathrm{depth}_{I(Z_i)} D_i \geq 1$ for $2 \leq i \leq c$ and $\mathrm{Hom}_{D_i}(M_i, M_i) = D_i$.*

Proof. If X is a standard determinantal scheme, defined by some $t \times (t + i - 1)$ matrix, and if we delete a column and let Y be the corresponding determinantal scheme, then Y is also standard determinantal [8]. Hence, if X is good determinantal, it follows that Y is also good determinantal by the definition of a good determinantal scheme. In particular, all X_i, $2 \leq i \leq c$, are good determinantal schemes and hence generic complete intersections. By the definition of Z_i, we get $\mathrm{depth}_{I(Z_i)} D_i \geq 1$.

Let $U_i = \mathrm{Proj}(D_i) - Z_i$ and note that $\tilde{M}_i|_{U_i}$ is an invertible sheaf. Let $S_r(M_i)$ be the rth symmetric power of the D_i-module M_i. For $1 \leq r \leq i-1$, $S_r(M_i)$ are maximal Cohen–Macaulay modules and $S_{i-1}(M_i)(\ell_i) \cong K_{D_i}(n+c+1)$ (cf. Proposition 2.2(3)). By (4.10) we have the injections

$$\mathrm{Hom}_{D_i}(S_r(M_i), S_r(M_i)) \hookrightarrow H^0_*(U_i, \mathcal{H}om(\widetilde{S_r(M_i)}, \widetilde{S_r(M_i)})) \cong H^0_*(U_i, \widetilde{D_i}).$$

Since $S_{i-1}(M_i)$ is a twist of the canonical module on D_i and since

$$\mathrm{Hom}(K_{D_i}, K_{D_i}) \cong D_i,$$

we get the lemma from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{D_i}(M_i, M_i) & \hookrightarrow & H^0_*(U_i, \widetilde{D_i}) \\ \psi \downarrow & & \| \\ \mathrm{Hom}_{D_i}(S_{i-1}(M_i), S_{i-1}(M_i)) & \hookrightarrow & H^0_*(U_i, \widetilde{D_i}). \end{array}$$

Indeed, ψ is injective and we conclude by

$$D_i \to \mathrm{Hom}_{D_i}(M_i, M_i) \hookrightarrow \mathrm{Hom}_{D_i}(S_{i-1}(M_i), S_{i-1}(M_i)) \cong D_i.$$ □

We are now ready to bound $\mathrm{aut}(\mathcal{B})$.

Proposition 4.2.5. *Assume $b_1 \leq \cdots \leq b_t$, $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$, and $c \geq 2$. Set $\ell := \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$. If $(c-1)a_{t+c-2} < \ell$, then $\mathrm{aut}(\mathcal{B}) = 1$. Otherwise, we have*

$$\mathrm{aut}(\mathcal{B}) \leq \sum_{i=1}^{c-3}\Bigg(\sum_{r+s=i} \sum_{\substack{0 \leq i_i < \cdots < i_r \leq t+i \\ 1 \leq j_1 \leq \cdots \leq j_s \leq t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} \cdots + b_{j_s}}{n+c}\Bigg)$$
$$+ \binom{h_0}{n+c} + 1,$$

where we set $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{c+t-3} + a_{c+t-2} - \ell + n + c$ *for* $i = 0, 1, \ldots, c-3$.

Proof. Set $S(\mathcal{B}) = \mathrm{Supp}(\mathcal{E}xt^1(\mathcal{B}, \mathcal{O}_{\mathbb{P}^{n+c}}))$. Since $\mathrm{codim}(S(\mathcal{B}), \mathbb{P}^{n+c}) \geq 3$ and $\mathrm{pd}(\mathcal{B}) = 1$, $\mathcal{B}$ is a rank $c-1$ reflexive sheaf on $\mathbb{P}^{n+c}$ ([74, Proposition 1.2]). Moreover, if $(c-1)a_{t+c-2} < \ell$ then, by [56, Lemma 10.1(ii)], $\mathcal{B}$ is stable and $\mathrm{aut}(\mathcal{B}) = 1$ because stable reflexive sheaves are simple. So, from now on, we assume $(c-1)a_{t+c-2} \geq \ell$ and we will proceed by induction on c by successively deleting columns from the right side, i.e., of the largest degree. For $c = 2$ the result was proved in [25] if $n \geq 1$ and in [56] for any $n \geq 0$. So, we will assume $c \geq 3$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) & = & \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^{n+c}}(b_i) & \xrightarrow{\varphi_c} & \oplus_{j=0}^{t+c-2}\mathcal{O}_{\mathbb{P}^{n+c}}(a_j) & \longrightarrow & \mathcal{B}_c & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^{n+c}}(b_i) & \xrightarrow{\varphi_{c-1}} & \oplus_{j=0}^{t+c-3}\mathcal{O}_{\mathbb{P}^{n+c}}(a_j) & \longrightarrow & \mathcal{B}_{c-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and the exact sequence

$$\begin{aligned}
0 &\longrightarrow \mathrm{Hom}(\mathcal{B}_{c-1}, \mathcal{B}_c) \longrightarrow \mathrm{Hom}(\mathcal{B}_c, \mathcal{B}_c) \\
&\xrightarrow{\alpha} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}), \mathcal{B}_c) = H^0(\mathcal{B}_c(-a_{t+c-2})) \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_{c-1}, \mathcal{B}_c) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_c, \mathcal{B}_c) \longrightarrow 0.
\end{aligned}$$

Moreover, if we tensor the following exact sequence with $. \otimes_R B_c$:

$$0 \to D_{c-1}(-a_{t+c-2}) \longrightarrow M_{c-1} \longrightarrow M_c \longrightarrow 0,$$

we get

$$\begin{aligned}
&\mathrm{Tor}_1^R(B_c, M_{c-1}) \longrightarrow \mathrm{Tor}_1^R(B_c, M_c) \longrightarrow D_{c-1}(-a_{t+c-2}) \otimes B_c \\
&\longrightarrow M_{c-1} \otimes B_c \cong \mathrm{Ext}^1_R(B_{c-1}, B_c) \longrightarrow M_c \otimes B_c \cong \mathrm{Ext}^1_R(B_c, B_c) \longrightarrow 0.
\end{aligned}$$

Applying Lemmas 4.2.3 and 4.2.4, we get

$$\mathrm{Tor}_1^R(B_c, M_{c-1}) = \mathrm{Hom}(M_c, M_{c-1}) = 0$$

(since M_c is supported in X_c which has codimension 1 in $X_{c-1} = \mathrm{Supp}(M_{c-1})$) and

$$\mathrm{Tor}_1^R(B_c, M_c) = \mathrm{Hom}(M_c, M_c) = D_c = A.$$

Hence,

$$H^0(\mathcal{B}_c(-a_{t+c-2})) \to \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{B}_{c-1},\mathcal{B}_c)$$

coincides with $(D_{c-1}(-a_{t+c-2})\otimes B_c)_0 \to (M_{c-1}\otimes B_c)_0$ whose kernel is $A_0 \cong K$, i.e., 1-dimensional. Therefore, $\dim(\operatorname{im}(\alpha)) = 1$ which gives us

$$\operatorname{aut}(\mathcal{B}_c) = \hom(\mathcal{B}_{c-1},\mathcal{B}_c) + 1. \tag{4.12}$$

We call $e \in \operatorname{Ext}^1(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}))$ the nontrivial extension ($e \neq 0$) satisfying

$$e: \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}) \longrightarrow \mathcal{B}_c \longrightarrow \mathcal{B}_{c-1} \longrightarrow 0.$$

We have

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2})) \longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{B}_c) \\ &\xrightarrow{\eta} \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{B}_{c-1}) \xrightarrow{\delta} \operatorname{Ext}^1(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2})). \end{aligned}$$

Since $\delta(1) = e \neq 0$, we have $\dim(\ker(\delta)) \leq \operatorname{aut}(\mathcal{B}_{c-1}) - 1$. On the other hand, using the hypothesis of induction to bound $\operatorname{aut}(\mathcal{B}_{c-1})$, we obtain

$$\begin{aligned} &\hom(\mathcal{B}_{c-1},\mathcal{B}_c) \\ &= \hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2})) + \dim(\operatorname{im}(\eta)) \\ &= \hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2})) + \dim(\ker(\delta)) \\ &\leq \hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2})) + \operatorname{aut}(\mathcal{B}_{c-1}) - 1 \\ &\leq \hom(\mathcal{B}_{c-1},\mathcal{O}(a_{t+c-2})) + 1 - 1 + \binom{h_0}{n+c} \qquad (4.13) \\ &\quad + \sum_{i=1}^{c-4}\left(\sum_{r+s=i}\ \sum_{\substack{0\leq i_1<\cdots<i_r\leq t+i \\ 1\leq j_1\leq\cdots\leq j_s\leq t}} (-1)^{i-r}\binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} \cdots + b_{j_s}}{n+c}\right), \end{aligned}$$

where we set $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{t+c-2} - \ell + n + c$ for all $i = 0,1,\ldots,c-3$. Now, we will compute $\hom(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2}))$. To this end, we first observe that $\operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}})$ is the first Buchsbaum–Rim module associated with

$$\varphi^*_{c-1}: G^*_{c-1} = \bigoplus_{j=0}^{t+c-3} R(-a_j) \longrightarrow F^* := \bigoplus_{i=1}^{t} R(-b_i).$$

Therefore, we have the following free graded R-resolution:

$$\begin{aligned} 0 &\longrightarrow \wedge^{t+c-2}G^*_{c-1} \otimes S_{c-3}(F) \otimes \wedge^t F \longrightarrow \cdots \\ &\longrightarrow \wedge^{t+i+1}G^*_{c-1} \otimes S_i(F) \otimes \wedge^t F \longrightarrow \cdots \\ &\longrightarrow \wedge^{t+1}G^*_{c-1} \otimes S_0(F) \otimes \wedge^t F \longrightarrow \operatorname{Hom}(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}) \longrightarrow 0. \end{aligned}$$

Since

$$\begin{aligned}\wedge^t F &= R\left(\sum_{i=1}^{t} b_i\right),\\ S_m(F) &= \bigoplus_{1\le j_1\le\cdots\le j_m\le t} R(b_{j_1}+\cdots+b_{j_m}),\\ \wedge^r(G^*_{c-1}) &= \bigoplus_{0\le i_1<\cdots<i_r\le t+c-3} R(-a_{i_1}-\cdots-a_{i_r}),\\ &= \bigoplus_{0\le i_1<\cdots<i_{t+c-2-r}\le t+c-3} R\left(-\sum_{j=0}^{t+c-3} a_j + a_{i_1}+\cdots+a_{i_{t+c-2-r}}\right),\\ &= \bigoplus_{0\le i_1<\cdots<i_{t+c-2-r}\le t+c-3} R\left(-\sum_{j=0}^{t+c-2} a_j + a_{t+c+2} + a_{i_1}+\cdots+a_{i_{t+c-2-r}}\right),\end{aligned}$$

we have

$$\begin{aligned}&\wedge^{t+i+1}(G^*_{c-1})\otimes S_i(F)\otimes\wedge^t F\\ &= \bigoplus_{\substack{0\le i_1<\cdots<i_{c-3-i}\le t+c-3\\ 1\le j_1\le\cdots\le j_i\le t}} R(-\ell+a_{t+c-2}+a_{i_1}+\cdots+a_{i_{c-3-i}}+b_{j_1}+\cdots+b_{j_i}).\end{aligned}$$

So,

$$\begin{aligned}&\dim_k(\wedge^{t+i+1}(G^*_{c-1})\otimes S_i(F)\otimes\wedge^t F)\\ &= \sum_{\substack{0\le i_1<\cdots<i_{c-3-i}\le t+c-3\\ 1\le j_1\le\cdots\le j_i\le t}} \binom{-\ell+a_{t+c-2}+a_{i_1}+\cdots+a_{i_{c-3-i}}+b_{j_1}+\cdots+b_{j_i}+n+c}{n+c},\end{aligned}$$

and we conclude that

$$\begin{aligned}&\hom(\mathcal{B}_{c-1},\mathcal{O}_{\mathbb{P}^{n+c}}(a_{t+c-2})) \qquad (4.14)\\ &= \sum_{r+s=c-3}\left(\sum_{\substack{0\le i_1<\cdots<i_r\le t+c-3\\ 1\le j_1\le\cdots\le j_s\le t}} (-1)^{c-3-r}\binom{h_{c-3}+a_{i_1}+\cdots+a_{i_r}+b_{j_1}+\cdots+b_{j_s}}{n+c}\right),\end{aligned}$$

where $h_{c-3} = 2a_{t+c-2} - \ell + n + c$. Putting altogether (4.12), (4.13), and (4.14), we obtain

$$\begin{aligned}\operatorname{aut}(\mathcal{B}_c) &\le \binom{h_0}{n+c} + 1\\ &+ \sum_{i=1}^{c-3}\left(\sum_{r+s=i}\sum_{\substack{0\le i_1<\cdots<i_r\le t+i\\ 1\le j_1\le\cdots\le j_s\le t}} (-1)^{i-r}\binom{h_i+a_{i_1}+\cdots+a_{i_r}+b_{j_1}\cdots+b_{j_s}}{n+c}\right),\end{aligned}$$

where we set $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{t+c-2} - \ell + n + c$ for $i = 0, 1, \ldots, c-3$, which proves Proposition 4.2.5. $\square$

Remark 4.2.6. Note that $\operatorname{aut}(\mathcal{B}) = 1$ provided $\ell > 2a_{t+c-2} + a_{t+c-3} + \cdots + a_{t+1}$ and $c > 3$. (Indeed all binomials in the expression in Proposition 4.2.5 vanish.)

We are now ready to give an upper bound for $\dim W(\underline{b}; \underline{a})$.

Theorem 4.2.7. *Assume* $a_0 \le a_1 \le \cdots \le a_{t+c-2}$, $b_1 \le \cdots \le b_t$, *and* $c \ge 2$. *Set* $\ell := \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$ *and* $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{t+c-2} - \ell + n + c$ *for* $i = 0, 1, \ldots, c-3$. *We have*

(1) *if* $(c-1)a_{t+c-2} < \ell$, *then*

$$\dim W(\underline{b}; \underline{a}) \le \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} - \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + 1;$$

(2) *if* $(c-1)a_{t+c-2} \ge \ell$, *then*

$$\dim W(\underline{b}; \underline{a}) \le \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} - \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + \binom{h_0}{n+c} + 1 + \sum_{i=1}^{c-3} \left(\sum_{r+s=i} \sum_{\substack{0 \le i_1 < \cdots < i_r \le t+i \\ 1 \le j_1 \le \cdots \le j_s \le t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} \cdots + b_{j_s}}{n+c} \right).$$

Proof. The theorem follows from inequality (4.4) and Proposition 4.2.5. $\square$

Remark 4.2.8. Note that if $c > 3$ and $\ell > 2a_{t+c-2} + a_{t+c-3} + \cdots + a_{t+1}$, then

$$\dim W(\underline{b}; \underline{a}) \le \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} - \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + 1.$$

Indeed, this follows from Theorem 4.2.7 and Remark 4.2.6.

Remark 4.2.9. Given integers $a_0, a_1, \ldots, a_{t+c-2}$ and $b_1, \ldots, b_t$, we always have $\dim W_s(\underline{b}; \underline{a}) = \dim W(\underline{b}; \underline{a})$. In fact, it is an easy consequence of Corollary 1.2.21 and the fact that a standard determinantal scheme is good determinantal if it is a generic complete intersection, and being a generic complete intersection is an open condition.

Example 4.2.10. (a) According to Ellingsrud's theorem ([25, Théorème 2]), in codimension 2 case, the bound given in Theorem 4.2.7 is sharp provided $n \geq 1$.

(b) According to [56, Proposition 1.12], in codimension 3 case, the bound given in Theorem 4.2.7 is sharp provided $n \geq 1$ and $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$.

(c) A *Rational normal scroll* $X \subset \mathbb{P}^N$ is a nondegenerate variety of minimal degree (i.e., $\deg(X) = \operatorname{codim}(X)+1$) defined by the maximal minors of a $2\times(c+1)$ matrix with linear entries ($c = \operatorname{codim}(X)$). As an example of rational normal scrolls we have the smooth, rational normal curves of degree d in $\mathbb{P}^d$. It is well known that the family of rational normal scrolls of degree d and codimension $d-1$ in $\mathbb{P}^N$ is irreducible of dimension $d(2N+2-d)-3$. So, again in this case, the bound given in Theorem 4.2.7 is sharp.

(d) Every closed subscheme $X \subset \mathbb{P}^{n+c}$ with Hilbert polynomial $p(t) = \binom{t+n}{n}$ is a linear space of dimension n and it is defined by c linear forms. Hence, $W(0;1,\ldots,1) = Gr(n+1, n+c+1) = \operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$. It is well known that the *Grassmannian*, $Gr(n+1, n+c+1)$, is a smooth, irreducible variety of dimension $c(n+1)$. So, again the bound given in Theorem 4.2.7 is sharp.

(e) Set $p(t) = d_1d_2d_3t - \frac{d_1d_2d_3(d_1+d_2+d_3-5)}{2}$. Let $W(d_1,d_2,d_3;0) \subset \operatorname{Hilb}^{p(t)}(\mathbb{P}^4)$ be the locus parameterizing complete intersection curves $C \subset \mathbb{P}^4$ of type (d_1,d_2,d_3). It is well known that $W(d_1,d_2,d_3;0)$ is irreducible of dimension

$$\sum_{i=1}^{3}\binom{d_i+4}{4} - \sum_{i,j}\binom{d_i-d_j+4}{4} + \binom{d_3-d_2-d_2+4}{4}.$$

Therefore, the bound given in Theorem 4.2.7 is sharp.

We are led to pose the following questions.

Question 4.2.11. *Let $W(\underline{b};\underline{a}) \subset \operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ (resp., $W_s(\underline{b};\underline{a})$) be the locus of good (resp., standard) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c defined by the maximal minors of a $t\times(t+c-1)$ matrix*

$$\mathcal{A} = (f_{ij})_{j=0,\ldots,t+c-2}^{i=1,\ldots,t},$$

where $f_{ij} \in K[x_0,x_1,\ldots,x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. We wonder

(1) *When is the closure of $W(\underline{b};\underline{a})$ an irreducible component of* $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$*?*

(2) *Is $W(\underline{b};\underline{a})$ smooth or, at least, generically smooth?*

(3) *Under which extra assumptions are the bounds given in Theorem* 4.2.7 *sharp?*

We will now address these questions. First, we will show that the inequality for $\operatorname{aut}(\mathcal{B})$ in Proposition 4.2.5 is indeed an equality. To this end, we will compute $\operatorname{aut}(\mathcal{B})$ by a different method, leading to an apparently new formula, and then prove that they coincide provided we assume that $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ and $b_1 \leq \cdots \leq b_t$ (see Proposition 4.2.14).

Lemma 4.2.12. *There is an exact sequence*

$$0 \to \mathrm{Hom}_R(B_c, F) \to \mathrm{Hom}_R(B_c, G_c) \to \mathrm{Hom}_R(B_c, B_c) \to \mathrm{Hom}_R(M_c, M_c) \to 0.$$

Proof. We apply $\mathrm{Hom}_R(B_c, .)$ to the exact sequence

$$0 \longrightarrow F \longrightarrow G_c \longrightarrow B_c \longrightarrow 0,$$

and we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(B_c, F) &\to \mathrm{Hom}_R(B_c, G_c) \to \mathrm{Hom}_R(B_c, B_c) \\ &\to \mathrm{Ext}^1_R(B_c, F) = M_c \otimes_R F \to \mathrm{Ext}^1(B_c, G_c) = M_c \otimes_R G_c. \end{aligned}$$

By Lemma 4.2.3, we have the exact sequence

$$0 \to \mathrm{Hom}_R(M_c, M_c) \to M_c \otimes_R F \to M_c \otimes_R G_c,$$

and we are done. □

Using the Buchsbaum–Rim resolution

$$\begin{aligned} 0 \longrightarrow \wedge^{t+c-1} G_c^* \otimes S_{c-2}(F) \otimes \wedge^t F \longrightarrow \cdots \longrightarrow \wedge^{t+i+1} G_c^* \otimes S_i(F) \otimes \wedge^t F \\ \longrightarrow \cdots \longrightarrow \wedge^{t+1} G_c^* \otimes S_0(F) \otimes \wedge^t F \longrightarrow \mathrm{Hom}(B_c, R) \longrightarrow 0, \end{aligned}$$

we immediately get the following corollary.

Corollary 4.2.13. *Set* $\tau_\nu := \hom_R(B_c, R)_\nu$. *Then,*

$$\mathrm{aut}(B_c) = 1 + \sum_{j=0}^{t+c-2} \tau_{a_j} - \sum_{i=1}^{t} \tau_{b_i}.$$

Proof. This follows from Lemmas 4.2.4 and 4.2.12 and the isomorphisms

$$\mathrm{Hom}_R(B_c, F) \cong \mathrm{Hom}_R(B_c, R) \otimes F \cong \oplus_{i=1}^t \mathrm{Hom}_R(B_c, R(b_i)).$$ □

Proposition 4.2.14. *Set* $K_i := \hom(B_{i-1}, R(a_{t+i-2}))_0$ *for* $3 \le i \le c$. *Suppose* $b_1 \le \cdots \le b_t$ *and* $a_0 \le a_1 \le \cdots \le a_{t+c-2}$. *Then we have*

$$\mathrm{aut}(\mathcal{B}) = 1 + K_3 + K_4 + \cdots + K_c,$$

and the inequality for $\mathrm{aut}(\mathcal{B})$ *in Proposition* 4.2.5 *is an equality.*

Proof. Dualizing the exact sequence $0 \to R(a_{t+c-2}) \to B_c \to B_{c-1} \to 0$, we get

$$0 \to \mathrm{Hom}(B_{c-1}, R) \to \mathrm{Hom}_R(B_c, R) \to R(-a_{t+c-2}) \to M_{c-1} \to M_c \to 0$$

which, together with the exact sequence (4.7), gives us the exact sequence

$$0 \to \mathrm{Hom}(B_{c-1}, R) \to \mathrm{Hom}_R(B_c, R) \to I_{D_{c-1}}(-a_{t+c-2}) \to 0. \tag{4.15}$$

Look at the commutative diagram

$$\begin{array}{ccc} & 0 & & 0 \\ & \downarrow & & \downarrow \\ & \mathrm{Hom}(B_{c-1},F)_0 & \longrightarrow & \mathrm{Hom}(B_{c-1},G_c)_0 \\ & \downarrow & & \downarrow \\ & \mathrm{Hom}(B_c,F)_0 & \longrightarrow & \mathrm{Hom}(B_c,G_c)_0 \\ & \downarrow & & \downarrow \\ 0= & \mathrm{Hom}(R(a_{t+c-2}),F)_0 & \longrightarrow & \mathrm{Hom}(R(a_{t+c-2}),G_c)_0. \end{array}$$

By (4.15), $\mathrm{Hom}(B_c,G_c)_0 \to \mathrm{Hom}(R(a_{t+c-2}),G_c)_0$ is zero because its image is $(I_{D_{c-1}} \otimes G_c(-a_{t+c-2}))_0$. Hence, we get

$$\begin{aligned} &\hom(B_c,G_c)_0 - \hom(B_c,F)_0 \\ &\quad = \hom(B_{c-1},G_{c-1})_0 + \hom(B_{c-1},R(a_{t+c-2}))_0 - \hom(B_{c-1},F)_0. \end{aligned}$$

Since we have

$$\mathrm{aut}(\mathcal{B}_c) = 1 + \hom(B_c,G_c)_0 - \hom(B_c,F)_0$$

by Lemma 4.2.12 and we may suppose

$$\mathrm{aut}(\mathcal{B}_{c-1}) = 1 + \hom(B_{c-1},G_{c-1})_0 - \hom(B_{c-1},F)_0,$$

we have proved

$$\mathrm{aut}(\mathcal{B}_c) = K_c + \mathrm{aut}(\mathcal{B}_{c-1}). \tag{4.16}$$

Now, we conclude by induction taking into account that

$$\mathrm{aut}(\mathcal{B}_2) = \hom(I_{D_2},I_{D_2})_0 = 1.$$

Moreover, combining equality (4.16) and the definition of K_c with equality (4.14), we see that the expression we have for $\mathrm{aut}(\mathcal{B}_c)$ coincides with the corresponding binomials in the expression of $\mathrm{aut}(\mathcal{B}_c)$ in Proposition 4.2.5, and it follows that the inequality must be an equality. □

So, we can rewrite Theorem 4.2.7 and we have the following proposition.

Proposition 4.2.15. *With the above notation,*

$$\begin{aligned} \dim W(\underline{b};\underline{a}) \le &\sum_{i,j}\binom{a_i-b_j+n+c}{n+c} - \sum_{i,j}\binom{a_i-a_j+n+c}{n+c} \\ &- \sum_{i,j}\binom{b_i-b_j+n+c}{n+c} + \sum_{j,i}\binom{b_j-a_i+n+c}{n+c} + 1 + K_3 + \cdots + K_c. \end{aligned}$$

Proof. This follows from inequality (4.4) and Proposition 4.2.14. □

Remark 4.2.16. One may show that the right-hand side of the inequality for $\dim W(\underline{b};\underline{a})$ in Proposition 4.2.15 is equal to $\dim \operatorname{Ext}^1_R(B_c, B_c)_0$. This indicates an interesting connection to the deformations of the R-module B_c.

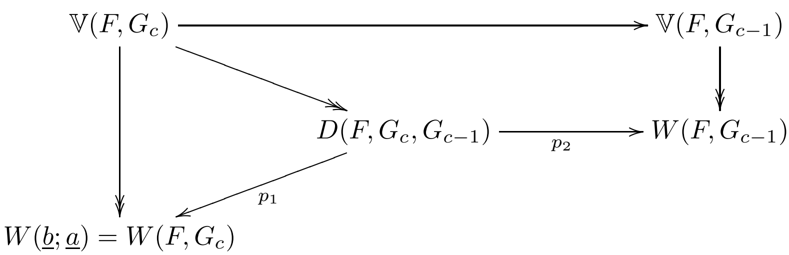
Our next purpose is to analyze when the bound given in Theorem 4.2.7 is sharp. We will see that, under mild conditions, the upper bound for $\dim W(\underline{b};\underline{a})$ given above is indeed the dimension of the standard determinantal locus $W(\underline{b};\underline{a})$ provided the codimension c is small. Indeed, if $2 \le c \le 3$ and $n \ge 1$, this is known (see [56], [25]). Using induction on c by successively deleting columns of the largest possible degree, we will see that for $4 \le c \le 5$, this is also the expected dimension and if $c \ge 6$, we also get the expected dimension formula for $W(\underline{b};\underline{a})$ under more restrictive assumptions.

If we denote by $W(F,G) := W(\underline{b};\underline{a})$ and by $\mathbb{V}(F,G_i) := \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\tilde{F},\tilde{G_i})$ the affine scheme whose rational points are the morphisms from $\tilde{F}$ to $\tilde{G_i}$, we have by the definition of $W(F,G_c)$ and $W(F,G_{c-1})$ a diagram of rational maps,

$$\begin{array}{ccc} \mathbb{V}(F,G_c) & \longrightarrow & \mathbb{V}(F,G_{c-1}) \\ \downarrow & & \downarrow \\ W(F,G_c) & & W(F,G_{c-1}), \end{array}$$

where the vertical down arrows are dominating and rational and where $\mathbb{V}(F,G_c) \to \mathbb{V}(F,G_{c-1})$ is defined by deleting the last column.

To prove that the upper bound of $\dim W(F,G_c)$ of Proposition 4.2.15 is also a lower bound, we need a deformation-theoretic technical result which computes the dimension of $W(F,G_c)$ in terms of the dimension of $W(F,G_{c-1})$. To do so, we consider the Hilbert flag scheme $D(p,q)$ parameterizing "pairs" $X \subset Y$ of closed subschemes of $\mathbb{P}^{n+c}$ with Hilbert polynomial p and q, respectively, and the subset $D(F,G_i,G_{i-1})$ of "pairs" $X \subset Y$, where $X \in W(F,G_i)$, is a good determinantal scheme defined by the matrix $\mathcal{A}_i \in \mathbb{V}(F,G_i)$, and $Y \in W(F,G_{i-1})$ is a good determinantal scheme defined by the matrix $\mathcal{A}_{i-1} \in \mathbb{V}(F,G_{i-1})$ obtained by deleting the last column of $\mathcal{A}_i$. Then the diagram above fits into



where p_1 and p_2 are the restrictions of the natural projections

$$pr_1 : D(p,q) \longrightarrow \operatorname{Hilb}^p(\mathbb{P}^{n+c}) \quad \text{and} \quad pr_2 : D(p,q) \longrightarrow \operatorname{Hilb}^q(\mathbb{P}^{n+c}),$$

respectively, and where $\mathbb{V}(F, G_c) \twoheadrightarrow D(F, G_c, G_{c-1})$ is dominating and rational by definition. Denoting

$$m_i(\nu) = \dim_K M_i(a_{t+i-2})_\nu,$$

we have the following proposition.

Proposition 4.2.17. *Let* $c \geq 3$. *Suppose* $W(\underline{b}; \underline{a}) \neq \emptyset$ *and* $\operatorname{depth}_{I(Z_{c-1})} D_{c-1} \geq 2$ *for a general* $D_{c-1} \in W(F; G_{c-1})$. *Then*

(1) p_2 *is dominating and* $\dim D(F, G_c, G_{c-1}) \geq \dim W(F, G_{c-1}) + m_c(0)$;
(2) $\dim W(\underline{b}; \underline{a}) \geq \dim W(F, G_{c-1}) + m_c(0) - \hom_{\mathcal{O}_{\mathbb{P}^{n+c}}}(\mathcal{I}_{X_{c-1}}, \mathcal{I}_{c-1})$.

Proof. Due to [61, Proposition 3.2], we see that for any $Y \in W(F, G_{c-1})$, there exists a regular section $R/I(Y) \hookrightarrow M_{c-1}(a_{t+c-2})$ whose cokernel is supported at some X with $\dim X < \dim Y$, and such that M_{c-1} is the cokernel of the morphism $\varphi^*_{c-1} : G^*_{c-1} \to F^*$. Moreover, for a given Y, the mapping cone construction shows that for any regular section $R/I(Y) \hookrightarrow M_{c-1}(a_{t+c-2})$, there is a morphism $\varphi^*_c : G^*_c \to F^*$ which reduces to the given φ^*_{c-1} by deleting the extra (say the last) column of the corresponding matrix. This shows that p_2 is dominating and that the fiber $p_2^{-1}(Y)$ "contains" the space of regular sections of $M_{c-1}(a_{t+c-2})$ in a natural way.

More precisely, note that every $Y \in W(F, G_{c-1})$ corresponds to some morphism φ_{c-1} between the same graded modules F and G_{c-1} . These modules determine all free graded modules in the Buchsbaum–Rim resolution $D_1(\varphi^*_{c-1})$ of $M_{c-1} = \operatorname{coker}(\varphi^*_{c-1})$ (cf. Proposition 1.2.16). Hence, for all $Y \in W(F, G_{c-1})$, the corresponding vector spaces $M_{c-1}(a_{t+c-2})_0$ have the same dimension. Since by (4.7) $M_c(a_{t+c-2})_0 = M_{c-1}(a_{t+c-2})_0/K$, it follows that $m_c(0)$ is the same number for all $Y \in W(F, G_{c-1})$. Now if $Y \in W(F, G_{c-1})$ is general, we have

$$\dim D(F, G_c, G_{c-1}) = \dim W(F, G_{c-1}) + \dim p_2^{-1}(Y)$$

by generic flatness. Hence, it suffices to see that $\dim p_2^{-1}(Y) \geq m_c(0)$. Pick $(X \subset Y) \in p_2^{-1}(Y)$, look at the exact sequence (4.9) and consider the injection $M_c(a_{t+c-2})_0 \hookrightarrow (N_{X/Y})_0$. In the tangent space $(N_{X/Y})_0$ of $pr_2^{-1}(Y) \supseteq p_2^{-1}(Y)$ at $(X \subset Y)$, we therefore have an $m_c(0)$-dimensional family arising from deforming the matrix $\mathcal{A} = [\mathcal{A}_{c-1}, L]$ of φ^*_c leaving φ^*_{c-1} (i.e., $\mathcal{A}_{c-1}$) fixed (L is the last column of $\mathcal{A}$). We may think of the last column of such a deformation of φ^*_c as $L + \sum_{i=1}^{m_c(0)} t_i L^{(i)} \mod (t_1, t_2, \ldots, t_{m_c(0)})^2$, where the t_i's are indeterminates and where the degree matrix of the columns $L^{(i)}$ are exactly the same as that of L. Since the degeneracy locus of the $t \times (t+c-1)$ matrix $[\mathcal{A}_{c-1}, L + \sum_{i=1}^{m_c(0)} t_i L^{(i)}]$ defines a flat family over some open subset T of $\operatorname{Spec}(K[t_1, \ldots, t_{m_c(0)}])$ containing the origin (because the Eagon–Northcott complex over $\operatorname{Spec}(K[\underline{t}])$ must be acyclic over some T provided the pullback to $(0) \in \operatorname{Spec}(K[\underline{t}])$ is acyclic), we see that the fiber $p_2^{-1}(Y)$ contains an $m_c(0)$-dimensional (linear) family, as required. Hence, p_2 is dominating and

$$\dim D(F, G_c, G_{c-1}) \geq \dim W(F, G_{c-1}) + m_c(0).$$

(2) It is straightforward to get (2) from (1). Indeed,

$$\dim D(F, G_c, G_{c-1}) - \dim W(\underline{b}; \underline{a}) \leq \dim p_1^{-1}(D_c),$$

and since $p_1^{-1}(D_c)$ is contained in the full fiber of the first projection $pr_1 : D(p,q) \to \mathrm{Hilb}^p(\mathbb{P}^{n+c})$ whose fiber dimension is known to have $\hom(\mathcal{I}_{X_{c-1}}, \mathcal{I}_{c-1})$ as an upper bound (see, e.g., [56, Chapter 9]), we easily conclude. □

Proposition 4.2.17 allows us, under some assumptions, to find a lower bound for $\dim W(\underline{b}; \underline{a})$ provided we have a lower bound for $\dim W(F, G_{c-1})$. Indeed, since it is easy to find $m_i(0)$ using the Buchsbaum–Rim resolutionof M_i, or by using the exact sequence (4.7) recursively, it remains to find $\hom(\mathcal{I}_{X_i}, \mathcal{I}_i)$ in terms of $\hom(\mathcal{I}_{X_{i-1}}, \mathcal{I}_{i-1})$.

Lemma 4.2.18. *Set* $a = a_{t+i-2} - a_{t+i-1}$.

(1) *If* $\mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} = 0$ *and* $\mathrm{depth}_{I(Z_{i-1})} D_{i-1} \geq 3$*, then*

$$\hom(I_{D_i}, I_i)_\nu \leq \dim(D_i)_{\nu+a} + \hom(I_{D_{i-1}}, I_{i-1})_{\nu+a}.$$

(2) *If* $\mathrm{Ext}^2_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} = 0$ *and* $\mathrm{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$*, then*

$$\mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_{i-1})_{\nu+a} = 0 \Rightarrow \mathrm{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_\nu = 0.$$

Remark 4.2.19. Since $I_{i-1} = M_{i-1}(a_{t+i-2})^*$, we have also

$$\mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}(a)) \cong \mathrm{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes M_{i-1}, M^*_{i-1}(-a_{t+i-2} - a_{t+i-1})).$$

Proof. (1) We consider the two exact sequences

$$0 \to \mathrm{Hom}_R(I_{i-1}, I_i) \to \mathrm{Hom}_R(I_{D_i}, I_i) \to \mathrm{Hom}_R(I_{D_{i-1}}, I_i), \tag{4.17}$$

$$0 \to \mathrm{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}) \to \mathrm{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_{i-1}) \to \mathrm{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i). \tag{4.18}$$

We have $\mathrm{depth}_{I(Z_{i-1})} D_{i-1} \geq 3$ and hence $\mathrm{depth}_{I(Z_{i-1})} D_i \geq 2$ and $\mathrm{depth}_{I(Z_{i-1})} I_i \geq 2$, and we get by (4.10)

$$\begin{aligned} \mathrm{Hom}(I_{i-1}, I_i) &\cong H^0_*(U_{i-1}, \mathcal{H}om(\mathcal{I}_{i-1}, \mathcal{I}_i)) \\ &\cong H^0_*(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i} \otimes \mathcal{I}^*_i, \mathcal{O}_{X_i})) \\ &\cong D_i(a) \end{aligned} \tag{4.19}$$

because, by the exact sequence (4.7),

$$\tilde{M}_{i-1}(a_{t+i-2}) \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}|_{U_{i-1}} \cong \tilde{M}_i(a_{t+i-1})(a)|_{U_{i-1}}$$

and hence

$$\mathcal{I}_{i-1} \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}|_{U_{i-1}} \cong \mathcal{I}_i(-a)|_{U_{i-1}}.$$

For similar reasons,

$$\begin{aligned}\operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I_{i-1}^*, D_{i-1}) &\cong H_*^0(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2 \otimes \mathcal{I}_{i-1}^*, \mathcal{O}_{X_{i-1}}))\\ &\cong H_*^0(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2, \mathcal{I}_{i-1}))\\ &\cong \operatorname{Hom}_R(I_{D_{i-1}}, I_{i-1}),\end{aligned}$$

and

$$\operatorname{Hom}_R(I_{D_{i-1}}, I_i) \cong H_*^0(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2 \otimes \mathcal{I}_i^*, \mathcal{O}_{X_i}))$$

is further isomorphic to

$$\begin{aligned}&\operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I_{i-1}^*, D_i(a))\\ &\quad\cong H_*^0(U_{i-1}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2 \otimes \mathcal{I}_{i-1}^* \otimes_{\mathcal{O}_{X_{i-1}}} \mathcal{O}_{X_i}, \mathcal{O}_{X_i}(a))).\end{aligned}$$

Putting all this together, we get that the exact sequences (4.17) and (4.18) reduce, in degree ν and $\nu + a$, respectively, to

$$\begin{aligned}0 \to (D_i)_{\nu+a} &\to \operatorname{Hom}_R(I_{D_i}, I_i)_\nu\\ &\to \operatorname{Hom}_R(I_{D_{i-1}}, I_i)_\nu \cong \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I_{i-1}^*, D_i)_{\nu+a},\end{aligned} \tag{4.20}$$

$$\begin{aligned}0 \to \operatorname{Hom}(I_{D_{i-1}} \otimes I_{i-1}^*, I_{i-1})_{\nu+a} &\to \operatorname{Hom}_R(I_{D_{i-1}}, I_{i-1})_{\nu+a}\\ &\to \operatorname{Hom}_{D_{i-1}}(I_{D_{i-1}} \otimes I_{i-1}^*, D_i)_{\nu+a} \to 0\end{aligned} \tag{4.21}$$

where the sequence (4.21) is short exact by assumption. Taking dimensions, we immediately get

$$\hom(I_{D_i}, I_i)_\nu \le \dim(D_i)_{\nu+a} + \hom(I_{D_{i-1}}, I_{i-1})_{\nu+a}$$

provided $\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I_{i-1}^*, I_{i-1})_{\nu+a} = 0$ and $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \ge 3$.

(2) As in (4.19), we see that

$$\operatorname{Ext}^1_{D_{i-1}}(I_{i-1}, I_i) \cong H_*^1(U_{i-1}, \mathcal{O}_{X_i}(a)) = 0. \tag{4.22}$$

Sheafifying the exact sequences (4.20) and (4.21) and taking global sections, we get

$$\begin{aligned}0 \to H_*^1(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_i}/\mathcal{I}_{X_i}^2, \mathcal{I}_i(-a))) &\to H_*^1(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2, \mathcal{I}_i(-a)))\\ &\qquad\qquad \|\\ \to H_*^1(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}_{i-1}^*, \mathcal{O}_{X_{i-1}})) &\to H_*^1(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}_{i-1}^*, \mathcal{O}_{X_i}))\\ &\to H_*^2(U_{i-1}, \mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}_{i-1}^*, \mathcal{I}_{i-1})).\end{aligned} \tag{4.23}$$

Since

$$\mathcal{H}om(\mathcal{I}_{X_{i-1}} \otimes \mathcal{I}_{i-1}^*, \mathcal{O}_{X_{i-1}}) \cong \mathcal{H}om(\mathcal{I}_{X_{i-1}}/\mathcal{I}_{X_{i-1}}^2, \mathcal{I}_{i-1}),$$

then $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$ and (4.10) show that the H_*^i-groups of (4.23) are isomorphic to the Ext^i-groups in the following diagram of exact horizontal sequences:

$$\begin{aligned} 0 \to \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i) &\to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i) \\ & \qquad \| \\ \to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_{i-1}(a)) &\to \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, D_i(a)) \\ &\to \operatorname{Ext}^2_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1}(a)) \end{aligned} \tag{4.24}$$

Using (4.24) we easily get that

$$\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_{i-1})_{\nu+a} = 0 \Rightarrow \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_\nu = 0$$

provided $\operatorname{Ext}^2_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_{\nu+a} = 0$ and $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$. □

Remark 4.2.20. By (4.20) the conclusion of Lemma 4.2.18 obviously holds provided we have $\operatorname{Hom}_R(I_{D_{i-1}}, I_i)_\nu = 0$. Using the Eagon–Northcott resolution of $I_{D_{i-1}}$ (i.e., of D_{i-1}), one may see that this Hom_ν-group vanishes if a_{t+i-2} is large enough.

Put

$$\begin{aligned} \lambda_c := \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} + \sum_{i,j} \binom{b_j - a_i + n + c}{n+c} \\ - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} - \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + 1, \end{aligned}$$

where the indices belonging to a_j (resp., b_i) ranges over $0 \leq j \leq t+c-2$ (resp., $1 \leq i \leq t$). We define λ_{c-1} by the analogous expression where now a_j (resp., b_i) ranges over $0 \leq j \leq t+c-3$ (resp., $1 \leq i \leq t$). It follows after a straightforward computation that

$$\begin{aligned} \lambda_c = \lambda_{c-1} + \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n + c}{n+c} \\ - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n+c} - \sum_{j=0}^{t+c-2} \binom{a_j - a_{t+c-2} + n + c}{n+c}. \end{aligned} \tag{4.25}$$

We will now show that the inequalities in Theorem 4.2.7 are equalities under certain assumptions. Recalling the equivalent expression of the upper bound of $\dim W(\underline{b};\underline{a})$ given in Proposition 4.2.15, we have the following theorem.

Theorem 4.2.21. *Let* $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ *and* $b_1 \leq \cdots \leq b_t$ *and assume* $a_{i-\min(2,t)} \geq b_i$ *for* $\min(2,t) \leq i \leq t$. *Let* $c \geq 3$ *and let* $W(\underline{b};\underline{a})$ *be the locus of good determinantal schemes in* $\mathbb{P}^{n+c}$, *where* $n \geq 0$ *if* $c \geq 4$ *and* $n \geq 1$ *if* $c = 3$.

For a general $\operatorname{Proj}(A) \in W(\underline{b}; \underline{a})$*, let* $R \twoheadrightarrow D_2 \twoheadrightarrow D_3 \twoheadrightarrow \cdots \twoheadrightarrow D_c = A$ *be the flag obtained by successively deleting columns from the right-hand side. If*

$$\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}} \otimes I^*_{i-1}, I_{i-1})_\nu = 0 \quad \text{for } \nu \le 0 \text{ and } 3 \le i \le c-1,$$

then

$$\dim W(\underline{b}; \underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c,$$

where $K_i = \hom(B_{i-1}, R(a_{t+i-2}))_0$ *for* $3 \le i \le c$.

Remark 4.2.22. If $c = 2$ and $n \ge 1$, one knows by [25] that

$$\dim W(\underline{b}; \underline{a}) = \lambda_2.$$

The same formula holds if $c = 2$ and $n = 0$ as well (see [26]). In this case one may get the formula by taking a general $\operatorname{Proj}(A) \in W(\underline{b}; \underline{a})$ and show that

$$\hom_R(I_A, A)_0 = \operatorname{ext}^1_R(I_A, I_A)_0 = \lambda_2$$

by, e.g., using [57, equation (26)].

Proof. Due to Remark 1.2.22 and the assumption $a_{i-\min(2,t)} \ge b_i$ for $\min(2,t) \le i \le t$, the set $Z_i = \operatorname{Sing}(X_i)$ satisfies $\operatorname{depth}_{I(Z_i)} D_i \ge 3$ for $2 \le i \le c-2$ and $\operatorname{depth}_{I(Z_{c-1})} D_{c-1} \ge 2$ (and also $\operatorname{depth}_{I(Z_2)} D_2 \ge 3$ in case $c = 3$ since $n \ge 1$) by choosing $X = \operatorname{Proj}(A)$ general in $W(\underline{b}; \underline{a})$.

To use Proposition 4.2.17, we only need to compute $\hom(I_{D_{c-1}}, I_{c-1})_0$ and $m_c(0)$ because we may, by induction, suppose that $\dim W(F, G_{c-1}) = \lambda_{c-1} + K_3 + \cdots + K_{c-1}$ for $c \ge 3$ (interpreting the expression as λ_2 when $c - 1 = 2$). By (4.7) and (4.6) we get

$$\begin{aligned} m_0(c) &= \dim M_{c-1}(a_{t+c-2})_0 - 1 \qquad (4.26)\\ &= \dim F^*(a_{t+c-2})_0 - \dim G^*_{c-1}(a_{t+c-2})_0 + \hom(B_{c-1}, R(a_{t+c-2}))_0 - 1\\ &= \sum_{i=1}^{t} \binom{a_{t+c-2} - b_i + n + c}{n+c} - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n+c} + K_c - 1. \end{aligned}$$

Thanks to Lemma 4.2.18, we can find an upper bound of $\hom(I_{D_{c-1}}, I_{c-1})_0$. We have

$$\hom(I_{D_{c-1}}, I_{c-1})_0 \le \binom{a+n+c}{n+c} + \hom(I_{D_{c-2}}, I_{c-2})_a$$

because $a = a_{t+c-3} - a_{t+c-2} \le 0$ and $\dim(D_i)_a$, which is either 0 or 1, must be equal to the binomial coefficient above. Repeated use of Lemma 4.2.18 implies

$$\hom(I_{D_{c-1}}, I_{c-1})_0 \le \sum_{j=t+1}^{t+c-3} \binom{a_j - a_{t+c-2} + n + c}{n+c} + \hom(I_{D_2}, I_2)_{a_{t+1} - a_{t+c-2}}. \qquad (4.27)$$

It remains to compute $\hom(I_{D_2}, I_2)_\alpha$ with $\alpha = a_{t+1} - a_{t+c-2}$. Using (4.10) (cf. the proof of Lemma 4.2.18), we get

$$\begin{aligned}\operatorname{Hom}(I_{D_2}, I_2) &\cong \operatorname{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_2)\\ &\cong \operatorname{Hom}_{D_2}(I_{D_2} \otimes M_2(a_{t+1}), D_2).\end{aligned}$$

Moreover, if $\ell_2 = \sum_{j=0}^t a_j - \sum_{i=1}^t b_i$, then $M_2 \cong K_{D_2}(-\ell_2+n+c+1)$ by Proposition 1.2.16. In codimension $c = 2$, one knows that

$$\begin{aligned}(I_{D_2}/I_{D_2}^2)^* &\cong \operatorname{Ext}_R^1(I_{D_2}, I_{D_2})\\ &\cong \operatorname{Ext}_R^1(I_{D_2}, D_2) \otimes I_{D_2}\\ &\cong K_{D_2}(n+c+1) \otimes I_{D_2},\end{aligned}$$

and since $\operatorname{depth}_{I(Z_2)} D_2 \geq 3$ and, hence, $\operatorname{depth}_{I(Z_2)} I_{D_2}/I_{D_2}^2 \geq 2$ (because the codepth of $I_{D_2}/I_{D_2}^2$ is less than or equal to 1 by [4]), we get

$$\begin{aligned}\operatorname{Hom}(I_{D_2}, I_2)_\alpha &\cong \operatorname{Hom}(I_{D_2} \otimes K_{D_2}(n+c+1), D_2)(\ell_2 - a_{t+1})_\alpha\\ &\cong (I_{D_2}/I_{D_2}^2)^{**}(\ell_2 - a_{t+1})_\alpha\\ &\cong (I_{D_2}/I_{D_2}^2)_{\ell_2 - a_{t+c-2}}.\end{aligned} \tag{4.28}$$

Thus the inequality $a_j \leq a_{t+c-2}$ and the exact sequences

$$\begin{aligned}&0 \to F \to G_2 = \oplus_{j=0}^t R(a_j) \to I_{D_2}(\ell_2) \to 0,\\ &0 \to \wedge^2 F \to F \otimes G_2 \to S_2 G_2 \to I_{D_2}^2(2\ell_2) \to 0\end{aligned} \tag{4.29}$$

show that

$$\hom(I_{D_2}, I_2)_\alpha = \dim(G_2)_{-a_{t+c-2}} = \sum_{j=0}^t \binom{a_j - a_{t+c-2} + n + c}{n+c}.$$

Using this last inequality together with (4.26), (4.27), and Proposition 4.2.17, we get by induction

$$\begin{aligned}&\dim W(\underline{b}; \underline{a})\\ &\geq \lambda_{c-1} + K_3 + \cdots + K_{c-1} + \sum_{i=1}^t \binom{a_{t+c-2} - b_i + n + c}{n+c}\\ &\quad - \sum_{j=0}^{t+c-3} \binom{a_{t+c-2} - a_j + n + c}{n+c} + K_c - 1 - \sum_{j=0}^{t+c-3} \binom{a_j - a_{t+c-2} + n + c}{n+c}\\ &= \lambda_c + K_3 + K_4 + \cdots + K_c,\end{aligned}$$

where the last equality is due to equality (4.25). Combining with Proposition 4.2.15, we get the theorem. □

Note that the vanishing assumption of Theorem 4.2.21 is empty if $c = 3$. Hence, we have the following corollary.

Corollary 4.2.23. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c = 3$. If $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$, then*

$$\dim W(\underline{b};\underline{a}) = \lambda_3 + K_3. \qquad \square$$

Remark 4.2.24. The above corollary essentially generalizes [56, Corollary 10.15(i)], where the depth condition is slightly stronger than the one we use in the proof of Theorem 4.2.21. The only missing part is that the assumption $n \geq 1$ excludes the interesting case of 0-dimensional good determinantal schemes. See Corollary 4.2.34 for the 0-dimensional case.

To apply Theorem 4.2.21 in the codimension $c = 4$ case, it suffices to prove that

$$\operatorname{Ext}^1_{D_2}(I_{D_2} \otimes I_2^*, I_2) = 0.$$

Due to Remark 4.2.19 and Proposition 1.2.16, the Ext^1-group above is isomorphic to a twist of

$$\operatorname{Ext}^1_{D_2}(I_{D_2} \otimes M_2, M_2^*) \cong \operatorname{Ext}^1_{D_2}(I_{D_2} \otimes K_{D_2}, K_{D_2}^*)(2\ell_2 - 2n - 2c - 2). \tag{4.30}$$

Hence, all we need follows from the following lemma.

Lemma 4.2.25. *Let $R \twoheadrightarrow D = R/I_D$ be a Cohen–Macaulay codimension 2 quotient and suppose $\operatorname{Proj}(D) \hookrightarrow \mathbb{P}^{n+c}$ is a local complete intersection outside a closed subset $Z \subset \operatorname{Proj}(D)$ which satisfies $\operatorname{depth}_{I(Z)} D \geq 4$. Then*

$$\operatorname{depth}_{\mathfrak{m}} \operatorname{Hom}_D(I_D \otimes K_D, K_D^*) \geq \operatorname{depth}_{\mathfrak{m}} D - 1.$$

In particular, $\operatorname{depth}_{I(Z)} \operatorname{Hom}_D(I_D \otimes K_D, K_D^*) \geq 3$ and hence

$$\operatorname{Ext}^1_D(I_D \otimes K_D, K_D^*) = 0.$$

Proof. D is standard determinantal, say $D = D_2$, and we have a minimal free R-resolution,

$$0 \to F \to G_2 \to I_D(\ell_2) \to 0 \tag{4.31}$$

as previously. If H_i is the ith Koszul homology built on some set of minimal generators of I_D, it suffices to show that there are two exact sequences,

$$0 \to \operatorname{Hom}_D(K_D(n+c+1), H_1) \to \wedge^2(F(-\ell_2)) \otimes D \to H_2 \to 0, \tag{4.32}$$

$$0 \to \operatorname{Hom}_D(K_D, H_1) \to K_D^* \otimes G_2(-\ell_2) \to \operatorname{Hom}(I_D \otimes K_D(n+c+1), K_D^*) \to 0. \tag{4.33}$$

Indeed, H_i are maximal Cohen–Macaulay modules by [50]. Hence, the first sequence shows that $\operatorname{Hom}_D(K_D(n+c+1), H_1)$ is maximal Cohen–Macaulay, while the second shows that the codepth of $\mathrm{Hom}(I_D \otimes K_D(n+c+1), K_D^*)$ is at most 1 and all conclusions of the lemma follow easily (cf. (4.10) for the last conclusion).

To see that (4.32) is exact we deduce, from (4.31), the exact sequence,

$$0 \to K_D(n+c+1)^* \to F(-\ell_2) \otimes_R D \to G_2(-\ell_2) \otimes_R D \to I_D/I_D^2 \to 0.$$

Indeed, we only need to prove that

$$K_D(n+c+1)^* = \ker[F(-\ell_2) \otimes_R D \to G_2(-\ell_2) \otimes_R D]$$

which follows by applying $\mathrm{Hom}_R(.,D)$ to the exact sequence

$$\cdots \to G_2(-\ell_2)^* \to F(-\ell_2)^* \to \mathrm{Ext}_R^1(I_D, R) \cong K_D(n+c+1) \to 0.$$

Since one, moreover, knows that

$$0 \to H_1 \to G_2(-\ell_2) \otimes_R D \to I_D/I_D^2 \to 0, \tag{4.34}$$

we get the exact sequence

$$0 \to K_D(n+c+1)^* \to F(-\ell_2) \otimes_R D \to H_1 \to 0, \tag{4.35}$$

from which we see that the Cohen–Macaulayness of $K_D(n+c+1)^*$ follows from that of H_1. Sheafifying the exact sequence (4.35) and using [39, Chapter II, Exec. 5.16], we get an exact sequence,

$$0 \to \tilde{K}_D(n+c+1)^* \otimes \tilde{H}_1|_U \to \wedge^2(\tilde{F}(-\ell_2)) \otimes \tilde{D}|_U \to \wedge^2 \tilde{H}_1|_U,$$

where $U = \mathrm{Proj}(D) - Z$. Applying $H_*^0(U,.)$ and recalling that $H_*^0(U, \wedge^2 \tilde{H}_1) \cong H_2$ (see [57, Proposition 18]), we get the exact sequence (4.32) because $\mathrm{depth}_{I(Z)} H_1 \geq 2$ implies $\mathrm{Hom}(K_D(n+c+1), H_1) \cong H_*^0(U, \tilde{K}_D(n+c+1)^* \otimes \tilde{H}_1)$ and the right most map in the exact sequence

$$0 \to \wedge^3(F(-\ell_2)) \to \wedge^3(G_2(-\ell_2)) \to \wedge^2(F(-\ell_2)) \to H_2 \to 0$$

(see [4]) must correspond to the map $\wedge^2(F(-\ell_2)) \otimes D \to H_2$ in (4.32) and the later is surjective (which one may prove directly as well, by applying $H_*^0(U, \tilde{K}_D^* \otimes (.))$ to (4.35), to get $H_*^0(U, \tilde{K}_D^* \otimes \tilde{H}_1) = 0$).

To see that (4.33) is exact, we dualize the exact sequence (4.34) and we get

$$0 \to (I_D/I_D^2)^* \to G_2(-\ell_2)^* \otimes D \to H_1^* \to 0$$

because $\mathrm{Ext}_D^1(I_D/I_D^2, D) \cong \mathrm{Ext}_D^1((I_D/I_D^2) \otimes K_D, K_D) = 0$ by the Cohen–Macaulayness of $(I_D/I_D^2) \otimes K_D(n+c+1) \cong (I_D/I_D^2)^*$, cf. the proof of Theorem 4.2.21 for the last isomorphism and [56, Chapter 6], for the Cohen–Macaulayness. Applying $\mathrm{Hom}_D(., K_D^*)$ to the last exact sequence we get the exact sequence (4.33) because $\mathrm{depth}_{I(Z)} D \geq 3$ implies $\mathrm{Hom}_D(H_1^*, K_D^*) \cong \mathrm{Hom}_D(K_D, H_1)$ and $\mathrm{Ext}_D^1(H_1^*, K_D^*) \cong H_*^1(U, \mathcal{H}om(\tilde{K}_D, \tilde{H}_1)) = 0$ where the vanishing is due to the Cohen–Macaulayness of $\mathrm{Hom}(K_D, H_1)$, which holds because we already have proved the exactness of (4.32). This concludes the proof. □

Corollary 4.2.26. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c = 4$. If $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$, then*

$$\dim W(\underline{b};\underline{a}) = \lambda_4 + K_3 + K_4.$$

Proof. Due to Remark 1.2.22 and the assumption $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$, when $X = \operatorname{Proj}(A)$ is chosen general in $W(\underline{b};\underline{a})$, then the set $Z_2 = \operatorname{Sing}(X_2)$ satisfies $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$. Hence, combining (4.30), Lemma 4.2.25, and Theorem 4.2.21, we are done. □

To apply Theorem 4.2.21 in the codimension $c = 5$ case, it suffices to prove that

$$\operatorname{Ext}^1_{D_3}(I_{D_3} \otimes I_3^*, I_3)_\nu = \operatorname{Ext}^1_{D_3}(I_{D_3} \otimes M_3, M_3^*(-a_{t+2} - a_{t+3}))_\nu = 0$$

for $\nu \leq 0$. Since $\operatorname{depth}_{I(Z_3)} D_3 \geq 3$ and I_3 is a maximal Cohen–Macaulay D_3-module, we have by (4.10)

$$\begin{aligned}\operatorname{Ext}^1_{D_3}(I_{D_3} \otimes M_3, M_3^*) &\cong H^1_*(U_3, \mathcal{H}om(\mathcal{I}_{X_3} \otimes \tilde{M}_3, \tilde{M}_3^*)) \\ &\cong H^1_*(U_3, \mathcal{H}om(\mathcal{I}_{X_3} \otimes S_2(\tilde{M}_3), \mathcal{O}_{X_3})) \\ &\cong \operatorname{Ext}^1_{D_3}(I_{D_3} \otimes S_2(M_3), D_3),\end{aligned}$$

where $U_3 = X_3 - Z_3$. Since, by Proposition 1.2.16(3), $K_{D_3}(n+c+1-\ell_3) \cong S_2(M_3)$ with $\ell_3 = \sum_{j=0}^{t+1} a_j - \sum_{i=1}^{t} b_i$, we get (letting $B := D_3$)

$$\begin{aligned}\operatorname{Ext}^1_{D_3}(I_{D_3} \otimes I_3^*, I_3) &= \operatorname{Ext}^1_B(I_B \otimes K_B(n+1+c-\ell_3), B(-a_{t+2} - a_{t+3})) \\ &\cong \operatorname{Ext}^1_B(I_B \otimes K_B(n+1+c), B)(\ell_3 - a_{t+2} - a_{t+3}).\end{aligned} \tag{4.36}$$

Lemma 4.2.27. *Let $R \to B = R/I_B$ be a codimension 3 good determinantal quotient, let $X \hookrightarrow \mathbb{P}^{n+c}$ be the corresponding embedding, and set $Z = \operatorname{Sing}(X)$.*

(1) *If $\operatorname{depth}_{I(Z)} B \geq 4$, then there is an exact sequence,*

$$0 \to \operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \to I_B/I_B^2 \to (I_B/I_B^2)^{**}$$

which preserves the grading. In particular,

(1a) $\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B)(\ell_3 - a_{t+2} - a_{t+3})_\nu = 0$ *for* $\nu < a_{t+3} + a_{t+2} - a_{t+1} - a_t$;

(1b) *if* $\operatorname{Char}(K) = 0$, *then* $\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B)(\ell_3 - a_{t+2} - a_{t+3})_\nu = 0$ *for* $\nu \leq a_{t+3} + a_{t+2} - a_{t+1} - a_t$.

(2) *If $\operatorname{depth}_{I(Z)} B \geq 5$, then there is an exact sequence,*

$$I_B/I_B^2 \to (I_B/I_B^2)^{**} \to \operatorname{Ext}^2_B(I_B \otimes K_B(n+c+1), B) \cong H^1_{I(Z)}(I_B/I_B^2) \to 0$$

which preserves the grading.

Remark 4.2.28. Note that (1b) shows the desired vanishing because in Theorem 4.2.21 we have assumed $a_0 \le a_1 \le \cdots \le a_{t+3}$.

Proof. (1) The Eagon–Northcott resolution associated with

$$\varphi_3 : F \to G_3 = \oplus_{j=0}^{t+1} R(a_j)$$

leads to

$$\begin{aligned} 0 \to F_3 &:= \wedge^{t+2} G_3^* \otimes S_2 F \otimes \wedge^t F \\ \to F_2 &:= \wedge^{t+1} G_3^* \otimes S_1 F \otimes \wedge^t F \\ \to F_1 &:= \wedge^t G_3^* \otimes \wedge^t F \to I_B \to 0. \end{aligned} \tag{4.37}$$

Applying $\operatorname{Hom}_R(.,R)$ we get the exact sequence

$$0 \to R \to F_1^* \to F_2^* \to F_3^* \to \operatorname{Ext}_R^2(I_B, R) \cong K_B(n+1+c) \to 0.$$

Tensoring with $. \otimes_R B$ leads to a complex,

$$0 \to (I_B/I_B^2)^* \to F_1^* \otimes B \to F_2^* \otimes B \xrightarrow{\psi} F_3^* \otimes B \to K_B(n+c+1) \to 0 \tag{4.38}$$

which is exact except in the middle where we have the homology $I_B \otimes K_B(n+c+1) \cong \operatorname{Tor}_1^R(K_B(n+c+1), B)$. Indeed this easily follows from the right exactness of $. \otimes_B B$ and the left exactness of $\operatorname{Hom}_R(., B)$ (applied to the exact sequence (4.37)). Tensoring with $. \otimes_R B$ the exact sequence (4.37) we get

$$0 \to H_1' := \ker(\rho) \to F_1 \otimes_R B \xrightarrow{\rho} I_B/I_B^2 \to 0 \tag{4.39}$$

(observe that ${H'}_1$ is quite close to the Koszul homology H_1). By [58, Lemma 35], we have $\operatorname{depth}_{\mathfrak{m}}(I_B/I_B^2)^* \ge \operatorname{depth}_{\mathfrak{m}} B - 1$ and hence by (4.10),

$$\operatorname{Ext}_B^1(I_B/I_B^2, B) = 0.$$

Dualizing the exact sequence (4.39), it follows that

$$0 \to (I_B/I_B^2)^* \to F_1^* \otimes B \to {H'}_1^* \to 0 \tag{4.40}$$

(and, if desirable, one may see $H_1^* \cong {H'}_1^*$). Since we know the homology "in the middle" of (4.38), we get the exact sequences

$$0 \to {H'}_1^* \to \ker(\psi) \to I_B \otimes K_B(n+c+1) \to 0, \tag{4.41}$$

$$0 \to \ker(\psi) \to F_2^* \otimes B \xrightarrow{\psi} F_3^* \otimes B \to K_B(n+c+1) \to 0. \tag{4.42}$$

Now we have the setup to prove that

$$\operatorname{Ext}_B^1(I_B \otimes K_B(n+c+1), B) \cong \mathcal{K} := \ker(I_B/I_B^2 \to (I_B/I_B^2)^{**}).$$

Firstly, note that by dualizing the exact sequence (4.40) once more and comparing with (4.39) in obvious way, we see that

$$0 \to {H'}_1 \to {H'}_1^{**} \to \mathcal{K} \to 0$$

by the snake lemma. Now we apply Hom(., B) to (4.41) and the left part of (4.42).

We get a commutative diagram,

$$\begin{array}{ccccccc} F_2 \otimes B & \to & {H'}_1 & \to & 0 & & \\ \downarrow & & \downarrow & & & & \\ \operatorname{Hom}(\ker(\psi), B) & \to & \operatorname{Hom}({H'}_1^*, B) & \to & \operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) & \to & \operatorname{Ext}^1_B(\ker(\psi), B) \\ \downarrow & & & & & & \\ \operatorname{Ext}^1_B(\operatorname{im}(\psi), B) & & & & & & \end{array}$$

from which we deduce the exact sequence

$$\begin{aligned} \operatorname{Ext}^1_B(\operatorname{im}(\psi), B) \to \mathcal{K} \to \operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \\ \to \operatorname{Ext}^1_B(\ker(\psi), B). \end{aligned} \tag{4.43}$$

Hence, it suffices to show that

$$\operatorname{Ext}^1_B(\operatorname{im}(\psi), B) = 0 = \operatorname{Ext}^1_B(\ker(\psi), B).$$

By (4.42) we have

$$\begin{aligned} \operatorname{Ext}^1_B(\operatorname{im}(\psi), B)(n+c+1) \cong \operatorname{Ext}^2_B(K_B, B) \cong \operatorname{Ext}^2_B(K_B \otimes K_B, K_B), \\ \operatorname{Ext}^1_B(\ker(\psi), B)(n+c+1) \cong \operatorname{Ext}^3_B(K_B, B) \cong \operatorname{Ext}^3_B(K_B \otimes K_B, K_B), \end{aligned}$$

where the rightmost isomorphism is a consequence of the spectral sequence used in [43, Satz 1.2] because we have $\operatorname{depth}_{I(Z)} B \geq 3$. By [13, Corollary 3.4], we know that $\operatorname{depth}_{\mathfrak{m}} S_2(K_B) \geq \operatorname{depth}_{\mathfrak{m}} B - 1$. Hence by Gorenstein duality

$$\operatorname{Ext}^i_B(S_2(K_B), K_B) = 0 \quad \text{for } i \geq 2.$$

Defining $\wedge$ by

$$0 \to \wedge \to K_B \otimes K_B \to S_2(K_B) \to 0$$

and noting that $\tilde{\wedge}|_{\operatorname{Proj}(B)-Z} = 0$, we get $\operatorname{Ext}^i_B(\wedge, K_B) = 0$ for $i \leq 3$ by (4.10) and the assumption $\operatorname{depth}_{I(Z)} B \geq 4$. Combining, we get $\operatorname{Ext}^i_B(K_B \otimes K_B, K_B) \cong \operatorname{Ext}^i_B(S_2(K_B), K_B) = 0$ for $i = 2$ and 3 as required; i.e., $\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \cong \mathcal{K}$ by (4.43).

Now it is a triviality to see (1a) because the smallest degree of a minimal generator of I_B is $\ell_3 - a_t - a_{t+1}$.

(1b) Since $\operatorname{depth}_{I(Z)} B \geq 2$, we get $(I_B/I_B^2)^{**} \cong H^0(X - Z, \mathcal{I}_X/\mathcal{I}_X^2)$ and hence $\mathcal{K}$ is isomorphic to $H^0_{I(Z)}(B)$. Similarly, we prove that the kernel of the "universal" derivation $d : I_B/I_B^2 \to \Omega_{R/K} \otimes_R B$ is $H^0_{I(Z)}(B)$ which, by [24, Theorem 3], is isomorphic to $I_B^{(2)}/I_B^2$ where $I_B^{(2)}$ is the second symbolic power of I_B. Hence we have a grading-preserving isomorphism,

$$\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B) \cong I_B^{(2)}/I_B^2. \tag{4.44}$$

Now, in characteristic zero, $I_B^{(2)} \subset \mathfrak{m} I_B$ by [24, Proposition 13], which shows that the smallest degree of the minimal generators of $I_B^{(2)}$ is at least one less than the smallest degree of the generators of I_B; i.e., we have

$$(I_B^{(2)})_{\ell_3 - a_{t+2} - a_{t+3} + \nu} = 0 \quad \text{for } \nu \leq 0,$$

and we conclude by (4.44).

(2) Again since $\operatorname{depth}_{I(Z)} B \geq 2$, we have $(I_B/I_B^2)^{**} \cong H^0_*(X - Z, \mathcal{I}_X/\mathcal{I}_X^2)$ and hence $\operatorname{coker}[I_B/I_B^2 \to (I_B/I_B^2)^{**}] \cong H^1_{I(Z)}(I_B/I_B^2) \cong H^2_{I(Z)}(H_1')$, cf. the exact sequence (4.39) for the last isomorphism. Using the exact sequence (4.41), we get the exact sequence

$$\begin{aligned} &\operatorname{Ext}^1_B(\ker(\psi), B) \to \operatorname{Ext}^1_B(H_1^{'*}, B) \\ &\qquad \to \operatorname{Ext}^2_B(I_B \otimes K_B(n+c+1), B) \to \operatorname{Ext}^2_B(\ker(\psi), B). \end{aligned}$$

As argued in (4.43) and after (4.43), we see that $\operatorname{Ext}^1_B(\ker(\psi), B) = 0$ and

$$\begin{aligned} \operatorname{Ext}^2_B(\ker(\psi), B)(n+c+1) &\cong \operatorname{Ext}^4_B(K_B \otimes K_B, K_B) \\ &\cong \operatorname{Ext}^4_B(S_2(K_B), K_B) = 0, \end{aligned}$$

where the last isomorphism to the second symmetric power follows from the fact that $\operatorname{depth}_{I(Z)} B \geq 5$ implies $\operatorname{Ext}^i_B(\wedge, K_B) = 0$ for $i \leq 4$, and the vanishing to the right follows from $\operatorname{depth}_{\mathfrak{m}} S_2(K_B) \geq \operatorname{depth}_{\mathfrak{m}} B - 1$. Since by (4.10),

$$\begin{aligned} \operatorname{Ext}^1_B(H_1^{'*}, B) &\cong H^1_*(U, \mathcal{H}om(\tilde{H}_1^{'*}, \tilde{B})) \\ &\cong H^1_*(U, \tilde{H}_1') \\ &\cong H^2_{I(Z)}(H_1'), \end{aligned}$$

we are done. □

Remark 4.2.29. For generic standard determinantal schemes, one knows by [10], that $\operatorname{depth}_{I(Z)}(I_B/I_B^2) \geq 2$. So the vanishing of $\operatorname{Ext}^1_B(I_B \otimes K_B(n+c+1), B)_\nu$ under reasonable genericity assumptions is expected (for any ν).

Corollary 4.2.30. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c = 5$. If $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$ and $\operatorname{Char}(K) = 0$, then*

$$\dim W(\underline{b};\underline{a}) = \lambda_5 + K_3 + K_4 + K_5.$$

Proof. It follows from Remark 1.2.22 and the assumption $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$ that the set $Z_j = \operatorname{Sing}(X_j)$ has $\operatorname{depth}_{I(Z_j)} D_j \geq 4$ for $j = 2$ and 3 provided $X = \operatorname{Proj}(A)$ is chosen general in $W(\underline{b};\underline{a})$. By (4.30), Lemma 4.2.25, (4.36), Lemma 4.2.27(1b), and Remark 4.2.28, the assumptions of Theorem 4.2.21 are fulfilled and we conclude by applying it. □

In the next corollary we show that the upper bound of $\dim W(\underline{b};\underline{a})$ given in Theorem 4.2.7 is indeed equal to $\dim W(\underline{b};\underline{a})$ for *all* $c \geq 3$ and most values of $a_0, a_1, \ldots, a_{t+c-2}; b_1, \ldots, b_t$. Our result is based upon Remark 4.2.20 and the proof of Theorem 4.2.21. Indeed, we have seen that

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + K_4 + \cdots + K_c$$

provided

$$\dim W(F, G_{c-1}) = \lambda_{c-1} + K_3 + K_4 + \cdots + K_{c-1}$$

and

$$\operatorname{Hom}_R(I_{D_{c-2}}, I_{c-1})_0 = 0. \tag{4.45}$$

Corollary 4.2.31. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 0$ and $c \geq 6$. Assume $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$, $\operatorname{Char}(K) = 0$, and*

(i$_6$): $a_{t+4} > a_{t-1} + a_t + a_{t+1} + a_{t+2} - a_0 - a_1 - a_2$,

(i$_7$): $a_{t+5} > a_{t-1} + a_t + a_{t+1} + a_{t+2} + a_{t+3} - a_0 - a_1 - a_2 - a_3$,

⋮

(i$_c$): $a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-4} a_j$.

Then,

$$\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + \cdots + K_c.$$

Proof. By the Eagon–Northcott resolution, the largest possible degree of a generator of $I_{D_{c-2}}$ is

$$\ell_c - \sum_{j=0}^{c-4} a_j - a_{t+c-3} - a_{t+c-2},$$

where $\ell_c = \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$ and the smallest possible degree of a generator of $I_{c-1} \cong I_{D_c}/I_{D_{c-1}}$ is

$$\ell_c - \sum_{j=t-1}^{t+c-3} a_j$$

because $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$. Hence if the latter is strictly larger than the former, i.e., if

$$a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-4} a_j,$$

then $\operatorname{Hom}(I_{D_{c-2}}, I_{c-1})_0 = 0$, and we conclude using the argument of (4.45) and Corollaries 4.2.23, 4.2.26, and 4.2.30. □

Remark 4.2.32. (1) If we want to skip the characteristic-zero assumption, we can avoid the use of Corollary 4.2.30 by introducing the assumption

(i$_5$): $\quad a_{t+3} > a_{t-1} + a_t + a_{t+1} - a_0 - a_1$.

We still get $\dim W(\underline{b};\underline{a}) = \lambda_c + K_3 + \cdots + K_c$, supposing $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$ and (i$_5$), (i$_6$),…,(i$_c$).

(2) We can further weaken

$$a_{i-\min(3,t)} \geq b_i \quad \text{for} \quad \min(3,t) \leq i \leq t$$

to

$$a_{i-\min(2,t)} \geq b_i \quad \text{for} \quad \min(2,t) \leq i \leq t$$

by avoiding the use of Corollary 4.2.26 and assuming in addition

(i$_4$): $\quad a_{t+2} > a_{t-1} + a_t - a_0$.

Remark 4.2.33. While Corollaries 4.2.23, 4.2.26, and 4.2.30 do not apply to the case when $W(\underline{b};\underline{a})$ is the locus of 0-dimensional determinantal schemes, Corollary 4.2.31 and Remark 4.2.32 do apply to the 0-dimensional case. In particular, using Remark 4.2.32(1) (resp., (2)) for $c = 5$ (resp., $c = 4$), we get a single assumption, namely (i$_5$) (resp., (i$_4$)) in addition to $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$ (resp., $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$) which suffices for having $\dim W(\underline{b};\underline{a})$ equal to the upper bound given in Theorem 4.2.7 for the zero schemes as well.

It is worthwhile to point out that this last remark on 0-dimensional schemes works also in the codimension $c = 3$ case, and here the (i$_3$) assumption is very weak. We have the following corollary.

Corollary 4.2.34. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+3}$ of codimension 3. If $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$ and if in addition*

(i$_3$): $\quad a_{t+1} > a_{t-1}$,

then $\dim W(\underline{b};\underline{a}) = \lambda_3 + K_3$.

Proof. Slightly extending Remark 1.2.22 by introducing the standard determinantal hypersurface $X_1 = \operatorname{Proj}(D_1)$, we have $\operatorname{depth}_{I(Z_1)} D_1 \geq 3$ and $\operatorname{depth}_{I(Z_2)} D_2 \geq 2$ by choosing $X = \operatorname{Proj}(A) \in W(\underline{b};\underline{a})$ general. It follows that (4.20) is exact also for $i = 1$, and since $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$ implies $\operatorname{Hom}_R(I_{D_1}, I_2)_0 = 0$, we get

$$\hom(I_{D_2}, I_2)_0 \cong \dim(D_2)_{a_t - a_{t+1}}.$$

Hence, Proposition 4.2.17(2) for $c = 3$ applies to explicitly get a lower bound of $\dim W(\underline{b};\underline{a})$, which, combining (4.26) and (4.25), turns out to be $\lambda_3 + K_3$. Hence, $\dim W(\underline{b};\underline{a}) = \lambda_3 + K_3$ by Theorem 4.2.7. □

Now we address the problem of when the closure of, $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ and when $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is smooth or, at least, generically smooth along $W(\underline{b};\underline{a})$. From now on, we assume $n \geq 1$ and $c \geq 2$. We have the following theorem.

Theorem 4.2.35. *Let $X \subset \mathbb{P}^{n+c}$ be a good determinantal scheme of dimension $n \geq 1$, let $X = X_c \subset X_{c-1} \subset \cdots \subset X_2 \subset \mathbb{P}^{n+c}$ be the flag obtained by successively deleting columns from the right-hand side, and let $Z_i \subset X_i$ be some closed subset such that $X_i - Z_i \subset \mathbb{P}^{n+c}$ is a local complete intersection. Let $p_i \in \mathbb{Q}[s]$ be the Hilbert polynomial of X_i.*

(1) *If $\operatorname{depth}_{I(Z_i)} D_i \geq 3$ for $2 \leq i \leq c-1$, and if $\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 \hookrightarrow \operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, D_i)_0$ for $i = 2, \ldots, c-1$, then X (and each X_i) is unobstructed, and*

$$\begin{aligned}&\dim_{X_{i+1}} \mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})\\ &= \dim_{X_i} \mathrm{Hilb}^{p_i}(\mathbb{P}^{n+c}) + \dim(N_{D_{i+1}/D_i})_0 - \hom(I_{D_i}, I_i)_0\end{aligned}$$

for $i = 2, 3, \ldots, c-1$.

(2) *If $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$, $b_1 \leq \cdots \leq b_t$, and $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$, and if a general $X \in W(\underline{b};\underline{a})$ satisfies*

$$\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0 \quad \text{for } i = 2, \ldots, c-1,$$

then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\mathrm{Hilb}^{p(s)}(\mathbb{P}^{n+c})$.

Proof. (1) First of all we claim that there are two short exact sequences, the vertical and the horizontal one, fitting into a commutative diagram (whose square is Cartesian)

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \operatorname{Hom}_R(I_{D_i}, I_i)_0 & & \\
 & & & & \downarrow & & \\
 & & A^1 & \xrightarrow{T_{pr_2}} & \operatorname{Hom}_R(I_{D_i}, D_i)_0 & & \\
 & & T_{pr_1}\downarrow & \square & \downarrow & & \\
0 \to \operatorname{Hom}(I_i, D_{i+1})_0 & \to & \operatorname{Hom}(I_{D_{i+1}}, D_{i+1})_0 & \to & \operatorname{Hom}_R(I_{D_i}, D_{i+1})_0 & \to 0 & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array} \tag{4.46}$$

where A^1 is the tangent space of the Hilbert flag scheme $D(p_{i+1}, p_i)$ at $(X_{i+1} \subset X_i)$ and T_{pr_i} are the tangent maps of the projections pr_i. Since the vertical sequence is exact by assumption and the tangent space description of the Hilbert flag scheme and its projections are well known (see [56, Chapter 6]), and note that the zero piece of the graded Hom's above and the corresponding global sections of their sheaves of [56] coincide by (4.10)), we only have to prove the short exactness of the horizontal sequence. Hence, it suffices to prove that T_{pr_2} is surjective. To see this, it suffices to slightly generalize the argument in the proof of Proposition 4.2.17, where we showed that the dimension of the fiber is greater than or equal $m_c(0)$. We skip the details since [56, Theorem 10.13] shows more. Indeed, it contains a deformation theoretic argument which shows that pr_2 is not only dominating but also "infinitesimal dominating or surjective" (i.e., smooth at $(X_{i+1} \subset X_i)$). In particular, we have that the tangent map T_{pr_2} is surjective (cf. Remark 4.2.36 for another argument).

By the proof of [56, Theorem 10.13], $D(p_{i+1}, p_i)$ is smooth at $(X_{i+1} \subset X_i)$ provided $\mathrm{Hilb}^{p_i}(\mathbb{P}^{n+c})$ is smooth at X_i (see Remark 4.2.36 for an easy argument). Since the tangent map of the first projection $pr_1 : D(p_{i+1}, p_i) \to \mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ is surjective, we get that $\mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ is smooth at X_{i+1}. By induction, X (and each X_i) is unobstructed since X_2 is unobstructed [25], and the two exact sequences of (4.46) easily lead to the dimension of $\dim_{X_{i+1}} \mathrm{Hilb}^{p_{i+1}}(\mathbb{P}^{n+c})$ because $N_{D_{i+1}/D_i} = \mathrm{Hom}(I_i, D_{i+1})$.

(2) To prove that $\overline{W(\underline{b}; \underline{a})}$ is an irreducible component, we use the notation of Proposition 4.2.17 and we may by induction suppose that $\overline{W(F, G_{c-1})}$ is an irreducible component of $\mathrm{Hilb}^{p_{c-1}}(\mathbb{P}^{n+c})$ since $\overline{W(F, G_2)}$ is an irreducible component by [25]. We have

$$\dim D(F, G_c, G_{c-1}) \geq \dim W(F, G_{c-1}) + m_c(0) \tag{4.47}$$

by Proposition 4.2.17(2), while for an irreducible component V of $D(p_c, p_{c-1})$ containing $D(F, G_c, G_{c-1})$, we must have

$$\dim V \leq \dim W(F, G_{c-1}) + \dim(N_{D_c/D_{c-1}})_0 \tag{4.48}$$

because $\dim(N_{D_c/D_{c-1}})_0$ is the fiber dimension of pr_2 at $(X_c \subset X_{c-1})$. Since $\operatorname{depth}_{I(Z_{c-1})} D_{c-1} \geq 3$, we have by (4.9)

$$\dim(N_{D_c/D_{c-1}})_0 = m_c(0).$$

Combining equalities (4.47) and (4.48) we get $\dim D(F, G_c, G_{c-1}) \geq \dim V$ and hence $\overline{D(F, G_c, G_{c-1})}$ is an irreducible component of $D(p_c, p_{c-1})$. Since the first projection $pr_1 : D(p_c, p_{c-1}) \to \mathrm{Hilb}^{p_c}(\mathbb{P}^{n+c})$ is smooth at $(X_{i+1} \subset X_i)$ by the surjectivity of T_{pr_1} and the smoothness of $D(p_{i+1}, p_i)$ at $(X_{i+1} \subset X_i)$, we get that $\overline{W(\underline{b}; \underline{a})}$ is an irreducible component, which necessarily is generically smooth because $\mathrm{Hilb}^{p_c}(\mathbb{P}^{n+c})$ is smooth at a general point X by the first part of the proof and by Remark 1.2.22. □

Remark 4.2.36. If, in Theorem 4.2.35, we suppose $\operatorname{depth}_{I(Z_i)} D_i \geq 4$, we may easily see the surjectivity of T_{pr_2} in the following way. Using (4.10), we get $\operatorname{Ext}^1_{D_i}(I_i,R_i)= \operatorname{Ext}^2_{D_i}(I_i,I_i)=0$ by the depth condition above. Applying $\operatorname{Hom}_{D_i}(I_i,.)$ to the exact sequence

$$0 \to I_i \to D_i \to D_{i+1} \to 0,$$

we get $\operatorname{Ext}^1_{D_i}(I_i, D_{i+1}) = 0$ and the lower horizontal sequence of (4.46) is short exact, and we easily conclude. Finally, using the vanishing of $\operatorname{Ext}^1_{D_i}(I_i, D_{i+1})_0$, it follows from (4.10) that $H^1(U_i, \tilde{N}_{D_{i+1}/D_i}) \cong \operatorname{Ext}^1_{D_{i+1}}(I_i/I_i^2, D_{i+1})_0 = 0$. Then it is not difficult to see that $D(p_{i+1}, p_i)$ is smooth at $(X_{i+1} \subset X_i)$ provided $\operatorname{Hilb}^{p_i}(\mathbb{P}^{n+c})$ is smooth at X_i.

To apply Theorem 4.2.35(2) in the codimension $c = 3$ case, it suffices to prove that

$$\operatorname{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2)_0 = 0.$$

By (4.10) and (4.28), we see that ($U_2 = X_2 - Z_2$)

$$\begin{aligned}\operatorname{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) &\cong H^1_*(U_2, \mathcal{H}om(\mathcal{I}_{X_2}, \tilde{K}_{D_2}(n+4), \mathcal{O}_{X_2})(\ell_2 - a_{t+1})) \\ &\cong H^1_*(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2}(\ell_2 - a_{t+1})).\end{aligned} \tag{4.49}$$

and we consider two cases.

(a) If $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$, we get $\operatorname{depth}_{I(Z_2)} I_{D_2}/I^2_{D_2} \geq 3$ (see [4]) and the group in (4.49) vanishes.

(b) If $\operatorname{depth}_{I(Z_2)} D_2 = 3$ (e.g., X_2 is smooth and 2-dimensional), the group, in degree zero, is clearly $H^1(U_2, \mathcal{I}_{X_2}/\mathcal{I}^2_{X_2}(\ell_2 - a_{t+1}))$, and we have to suppose that it vanishes in order to conclude that $\overline{W(\underline{b};\underline{a})}$ is a generically smooth irreducible component of $\operatorname{Hilb}^p(\mathbb{P}^{n+3})$ of dimension $\lambda_3 + K_3$. All this is essentially [56, Corollary 10.15 (ii)].

The case $c = 4$ is also straightforward. In this case it suffices to see that (4.49) vanishes and that

$$\operatorname{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3) = 0. \tag{4.50}$$

If we suppose

$$\operatorname{depth}_{I(Z_2)} D_2 \geq 4 \quad \text{and} \quad \operatorname{depth}_{I(Z_3)} D_3 \geq 4 \tag{4.51}$$

we claim that both groups vanish. We only need to prove (4.50).

Since $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$, it follows from (4.10) that (4.20) is short exact for $i = 3$. Using Lemma 4.2.25 and (4.30), we see that (4.21) is short-exact for $i = 3$ as well; i.e., we have exact sequences

$$\begin{array}{c} 0 \to D_3(a) \to \operatorname{Hom}_R(I_{D_3}, I_3) \to \operatorname{Hom}_R(I_{D_2}, I_3) \to 0 \qquad (4.52) \\ \| \\ 0 \to \operatorname{Hom}(I_{D_2} \otimes I_2^*, I_2(a)) \to \operatorname{Hom}_R(I_{D_2}, I_2(a)) \to \operatorname{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_3(a)) \to 0,\end{array}$$

where $a = a_{t+1} - a_{t+2}$. By Lemma 4.2.25, the codepth of $\operatorname{Hom}(I_{D_2} \otimes I_2^*, I_2(a))$ is at most 1, while (4.28) shows the same conclusion for $\operatorname{Hom}(I_{D_2}, I_2(a))$. The lower exact sequence of (4.52) therefore shows that the codepth of $\operatorname{Hom}_{D_2}(I_{D_2} \otimes I_2^*, D_3(a))$ is at most 1 as a D_3-module. The upper sequence shows that

$$\operatorname{depth}_{\mathfrak{m}} \operatorname{Hom}_R(I_{D_3}, I_3) \geq \operatorname{depth}_{\mathfrak{m}} D_3 - 1. \tag{4.53}$$

Now since $\operatorname{depth}_{I(Z_3)} D_3 \geq 4$, we get $\operatorname{depth}_{I(Z_3)} \operatorname{Hom}_{D_3}(I_{D_3}/I_{D_3}^2, I_3) \geq 3$ and hence by (4.10) we get that (4.50) holds. By Remark 1.2.22, we see that (4.51) holds for a general $X \in W(\underline{b}; \underline{a})$ provided $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$. Combining with Corollaries 4.2.23 and 4.2.26, we get the following.

Corollary 4.2.37. *Let $W(\underline{b}; \underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 2$ and $c = 3$ or $c = 4$. If $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$, then $\overline{W(\underline{b}; \underline{a})}$ is a generically smooth, irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ of dimension $\lambda_c + K_3 + \cdots + K_c$.*

Remark 4.2.38. If $c = 4$, then the assumption (4.51) excludes the interesting case when $W(\underline{b}; \underline{a})$ parameterizes good determinantal curves in $\mathbb{P}^5$. To consider this case we will weaken (4.51) and only suppose $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$. Recalling that (4.28) leads to

$$H_*^1(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{O}_{X_2})) \cong H_*^1(U_2, \mathcal{I}_{X_2}/\mathcal{I}_{X_2}^2(\ell_2 - a_{t+1})) = 0,$$

we have by (4.23) the injections

$$\begin{array}{r} \operatorname{Ext}^1_{D_3}(I_{D_3}/I_{D_3}^2, I_3)_0 \hookrightarrow H^1(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{O}_{X_3}(a))) \hookrightarrow H^2(U_2, \mathcal{H}om(\mathcal{I}_{X_2} \otimes \mathcal{I}_2^*, \mathcal{I}_2(a))) \\ \| \qquad\qquad\qquad\qquad\qquad\qquad \| \\ H^1(U_2, \mathcal{I}_{X_2}/\mathcal{I}_{X_2}^2 \otimes \mathcal{O}_{X_3}(\ell_2 - a_{t+2})) \hookrightarrow H^2(U_2, \mathcal{I}_{X_2}/\mathcal{I}_{X_2}^2 \otimes \tilde{K}_{D_2}^*(a')), \end{array} \tag{4.54}$$

where $a = a_{t+1} - a_{t+2}$ and $a' = 2\ell_2 - 6 - a_{t+1} - a_{t+2}$. In the interesting case $X = X_4 \subset X_3 \subset X_2 \subset \mathbb{P}^5$ where X_2 is smooth, then $U_2 = X_2$. In particular, if one of the groups of (4.54) vanishes, then $\overline{W(\underline{b}; \underline{a})}$ is still a generically smooth, irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^5)$ of dimension $\lambda_4 + K_3 + K_4$.

As a corollary of the first part of Theorem 4.2.35, we also get the unobstructedness and the vanishing of some $H_*^i(X, \mathcal{N}_X)$ for good determinantal schemes $X \in \mathbb{P}^{n+c}$ of codimension $3 \leq c \leq 4$. For $c = 3$, the unobstructedness is essentially proved by [56, Corollary 10.15] and the vanishing of $H_*^i(X, \mathcal{N}_X)$ is shown in [58, Lemma 35]. Notice that the corollary really gives additional information about the generically smooth component $\overline{W(b; a)}$ of Corollary 4.2.37, because it tells more precisely where $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is smooth. Finally, note also that there exists obstructed good determinantal reduced curves in $\mathbb{P}^4$, cf. [56, Remark 9.12], so some kind of limitations on the singular locus of X is expected to get unobstructedness.

Corollary 4.2.39. *Let* $X = \operatorname{Proj}(A) \subset \mathbb{P}^{n+c}$ *be a good determinantal scheme of dimension* $n \geq 2$ *for which there is a flag satisfying* $\operatorname{depth}_{I(Z_i)}(D_i) \geq 4$ *for* $2 \leq i \leq c-1$. *If* $3 \leq c \leq 4$, *then* X *is unobstructed and the normal module* $N_A := \operatorname{Hom}_R(I_A, A)$ *satisfies* $\operatorname{depth}_{\mathfrak{m}}(N_A) \geq n-1$. *In particular,*

$$H^i_*(X, \mathcal{N}_X) = 0 \quad \text{for } 1 \leq i \leq n-2.$$

Proof. Due to the vanishing of (4.49) and (4.50), the unobstructedness of X follows at once from Theorem 4.2.35(1). Moreover, exactly as we managed to show that the exact sequences of (4.52) implied (4.53), we may see that the graded exact sequences

$$0 \to \operatorname{Hom}_R(I_{D_2}, I_2) \to \operatorname{Hom}_R(I_{D_2}, D_2) \to \operatorname{Hom}_R(I_{D_2}, D_3) \to 0$$

and

$$0 \to \operatorname{Hom}(I_2, D_3) \to \operatorname{Hom}(I_{D_3}, D_3) \to \operatorname{Hom}_R(I_{D_2}, D_3) \to 0$$

imply

$$\operatorname{depth}_{\mathfrak{m}} \operatorname{Hom}(I_{D_3}, D_3) \geq \dim D_3 - 1, \tag{4.55}$$

because we have $\operatorname{depth}_{I(Z_2)}(D_2) \geq 4$ and hence $\operatorname{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2) = 0$ by (4.49). This shows what we want for $c = 3$. Finally, the same argument for $i = 3$, assuming (4.51) and using (4.53) and (4.55), leads to $\operatorname{depth}_{\mathfrak{m}} \operatorname{Hom}_R(I_{D_4}, D_4) \geq \dim D_4 - 1$ and we are done. □

In the following example we will see that Corollary 4.2.37 does not always extend to good determinantal curves $C \subset \mathbb{P}^5$; i.e., the closure of $W(\underline{b}; \underline{a})$ is not necessarily an irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^5)$, although by Corollary 4.2.26 we know that $\dim W(\underline{b}; \underline{a})$ is indeed $\lambda_4 + K_3 + K_4$.

Example 4.2.40. Let $C \subset \mathbb{P}^5$ be a smooth good determinantal curve of degree 15 and arithmetic genus 10 defined by the maximal minors of a 3×6 matrix with linear entries. The closure of $W(\underline{b}; \underline{a}) = W(0,0,0;1,1,1,1,1,1)$ inside $\operatorname{Hilb}^{15t-9}(\mathbb{P}^5)$ is not an irreducible component. In fact, let $H_{15,10} \subset \operatorname{Hilb}^{15t-9}(\mathbb{P}^5)$ be the open subset parameterizing smooth connected curves of degree $d = 15$ and arithmetic genus $g = 10$. It is well known that any irreducible component of $H_{15,10}$ has dimension $\geq \chi(N_C) = 6d + 2(1-g) = 72$ (cf. [81, Section 11b]); while by Corollary 4.2.26, $\dim W(\underline{b}; \underline{a}) = 64$.

For the codimension $c = 5$ case, we have the following corollary.

Corollary 4.2.41. *Let* $W(\underline{b}; \underline{a})$ *be the locus of good determinantal schemes in* $\mathbb{P}^{n+c}$ *where* $n \geq 1$ *and* $c = 5$. *If* $a_{i-\min(3,t)} \geq b_i$ *for* $\min(3,t) \leq i \leq t$ *and if* $W(\underline{b}; \underline{a})$ *contains a determinantal scheme* $X = \operatorname{Proj}(D_5)$ *whose flag* $R \to D_2 \to D_3 \to D_4 \to D_5$, *obtained by deleting columns of "largest possible degree", satisfies (with* $Z_i = \operatorname{Sing}(X_i)$) $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$, $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$, *and* $H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3})) = 0$, *then* $\overline{W(\underline{b}; \underline{a})}$ *is a generically smooth, irreducible component of* $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+5})$ (*of dimension* $\lambda_5 + K_3 + K_4 + K_5$ *provided* $\operatorname{Char}(K) = 0$).

Proof. First of all, note that by Remark 1.2.22, if we choose $X \in W(\underline{b};\underline{a})$ general, we have $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$ and $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$. By Theorem 4.2.35 and the conclusion of (4.51), it suffices to show that

$$\operatorname{Ext}^1_{D_4}(I_{D_4}/I^2_{D_4}, I_4)_0 = 0. \tag{4.56}$$

By Lemma 4.2.18(2) and (4.50), we must show that

$$\operatorname{Ext}^2_{D_3}(I_{D_3} \otimes I_3^*, I_3)_{a_{t+2}-a_{t+3}} = 0.$$

Looking at (4.36), this group is isomorphic to

$$\operatorname{Ext}^2_{D_3}(I_{D_3} \otimes K_{D_3}(n+c+1), D_3)_{\ell_3-2a_{t+3}}$$

which by Lemma 4.2.27(2) is further isomorphic to $H^1_{I(Z_3)}(I_{D_3}/I^2_{D_3})_{\ell_3-2a_{t+3}}$. Since $X_3 = \operatorname{Proj}(D_3)$ is Cohen–Macaulay, the cohomology sequence associated with

$$0 \to I^2_{D_3} \to I_{D_3} \to I_{D_3}/I^2_{D_3} \to 0$$

gives us

$$\begin{aligned} H^1_{I(Z_3)}(I_{D_3}/I^2_{D_3})_{\ell_3-2a_{t+3}} &\cong H^2_{I(Z_3)}(I^2_{D_3})_{\ell_3-2a_{t+3}} \\ &\cong H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3})), \end{aligned}$$

and we get (4.56). □

We will now give two examples. The first one will be a smooth standard determinantal surface $S \subset \mathbb{P}^7$ whose flag $R \to D_2 \to D_3 \to D_4 \to D_5$, obtained by deleting columns of "largest possible degree", satisfies all hypotheses required in Corollary 4.2.41 and hence $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^7)$ of dimension $\lambda_5 + K_3 + K_4 + K_5$ ($\operatorname{Char}(K) = 0$). The second one will be a smooth standard determinantal curve $C \subset \mathbb{P}^6$, hence the condition $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$ is not fulfilled and in this case we will see that the closure of $W(\underline{b};\underline{a})$ is not an irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^6)$, although by Corollary 4.2.30 we know that $\dim W(\underline{b};\underline{a})$ is indeed $\lambda_5 + K_3 + K_4 + K_5$ ($\operatorname{Char}(K) = 0$).

Example 4.2.42. (a) Let $S \subset \mathbb{P}^7$ be a smooth good determinantal surface of degree 6 defined by the maximal minors of a 2×6 matrix with general linear entries. Let $R \to D_2 \to D_3 \to D_4 \to D_5$ be the flag obtained by deleting columns from the right-hand side. With the computer program Macaulay 2 [34] we check that all hypotheses required in Corollary 4.2.41 are satisfied; i.e., $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$, $\operatorname{depth}_{I(Z_3)} D_3 \geq 5$ and $H^1(X_3 - Z_3, \mathcal{I}^2_{X_3}(\ell_3 - 2a_{t+3})) = H^1(X_3, \mathcal{I}^2_{X_3}(2)) = 0$. By Corollary 4.2.41, the closure of $W(\underline{b};\underline{a})$ inside $\operatorname{Hilb}^{p(t)}(\mathbb{P}^7)$ is a generically smooth, irreducible component of dimension 57.

(b) Let $C \subset \mathbb{P}^6$ be a smooth good determinantal curve of degree 21 and arithmetic genus 15 defined by the maximal minors of a 3×7 matrix with linear entries. Since $\dim(C) = 1$, we have $\dim(X_3) = 3$, and hence $\operatorname{depth}_{I(Z_3)} D_3 \leq 4$. The closure of $W(\underline{b};\underline{a})$ inside $\operatorname{Hilb}^{21t-14}(\mathbb{P}^6)$ is not an irreducible component. In fact, let $H_{21,15} \subset \operatorname{Hilb}^{21t-14}(\mathbb{P}^6)$ be the open subset parameterizing smooth connected curves of degree $d = 21$ and arithmetic genus $g = 15$. It is well known that any irreducible component of $H_{21,15}$ has dimension $\geq 7d + 3(1-g) = 105$; while by Corollary 4.2.30, $\dim W(0,0,0;1,1,1,1,1,1,1) = 90$.

Our final corollaries are similar to Corollaries 4.2.31 and 4.2.34. To apply the final part of Theorem 4.2.35, we must show that $\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0$ for $i = 2, \ldots, c-1$. Using, however, the upper sequence of (4.24) it suffices to show that $\operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i)_0 = 0$ provided $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \geq 4$. This vanishing is fulfilled if

$$\operatorname{Ext}^1_R(I_{D_{i-1}}, I_i)_0 = 0 \quad \text{for } i = 4, \ldots, c-1 \tag{4.57}$$

(since $\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0$ for $i = 2, 3$ provided $\dim_{I(Z_i)} D_i \geq 4$ by (4.49) and (4.50)).

Corollary 4.2.43. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ where $n \geq 1$ and $c \geq 5$, or $n \geq 2$ and $3 \leq c \leq 4$. Assume $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$. Moreover, if $c \geq 5$, then*

(j5): $a_{t+3} > a_{t-1} + a_t + a_{t+1} - a_0 - b_1$,

(j6): $a_{t+4} > a_{t-1} + a_t + a_{t+1} + a_{t+2} - a_0 - a_1 - b_1$,

⋮

(jc): $a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-5} a_j - b_1$.

Then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ of dimension $\lambda_c + K_3 + \cdots + K_c$.

Proof. The relation of $I_{D_{c-2}}$ of the largest possible degree is

$$\ell_c - \sum_{j=0}^{c-5} a_j - b_1 - a_{t+c-3} - a_{t+c-2},$$

where $\ell_c = \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$ and the smallest possible degree of a generator of I_{c-1} is

$$\ell_c - \sum_{j=t-1}^{t+c-3} a_j.$$

Hence, $\operatorname{Ext}^1_R(I_{D_{c-2}}, I_{c-1})_0 = 0$ if

$$\ell_c - \sum_{j=0}^{c-5} a_j - b_1 - a_{t+c-3} - a_{t+c-2} < \ell_c - \sum_{j=t-1}^{t+c-3} a_j$$

or, equivalently,

$$a_{t+c-2} > \sum_{j=t-1}^{t+c-4} a_j - \sum_{j=0}^{c-5} a_j - b_1$$

which is our assumption ($\mathrm{j_c}$).

Similarly, we get $\operatorname{Ext}^1_R(I_{D_{i-1}}, I_i)_0 = 0$ if ($\mathrm{j_{i+1}}$) holds. Since, by Remark 1.2.22 and the hypothesis $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$, we know that a general $X \in W(\underline{b};\underline{a})$ satisfies $\operatorname{depth}_{I(Z_i)} D_i \geq 4$ for $2 \leq i \leq c-2$, we conclude by (4.57). For the dimension formula we use Remark 4.2.32(1). □

Since Corollary 4.2.43 does not apply to $n = 1$ and $3 \leq c \leq 4$, we include one more result to cover these cases. For $c = 3$, the result is known (see [56, Corollary 10.11]).

Corollary 4.2.44. *Let $W(\underline{b};\underline{a})$ be the locus of good determinantal schemes in $\mathbb{P}^{n+c}$ of dimension $n \geq 1$. If either*

(1) *$c = 3$, $a_{i-\min(2,t)} \geq b_i$ for $\min(2,t) \leq i \leq t$ and $a_{t+1} > a_{t-1} + a_t - b_1$, or*
(2) *$c = 4$, $a_{i-\min(3,t)} \geq b_i$ for $\min(3,t) \leq i \leq t$ and $a_{t+2} > a_{t-1} + a_t - b_1$,*

then $\overline{W(\underline{b};\underline{a})}$ is a generically smooth, irreducible component of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ of dimension $\lambda_c + K_3 + \cdots + K_c$.

Proof. Let $c = 3$. To see the vanishing of $\operatorname{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_2)_0$ of Theorem 4.2.35, it suffices to prove that $\operatorname{Ext}^1_R(I_{D_2}, I_2)_0 = 0$. As in the proof of Corollary 4.2.43, we find the minimal degree of relations of I_{D_2} to be $\ell_2 - b_1$, and we get the vanishing of the Ext^1_R-group above by assuming $\ell_2 - b_1 = \ell_3 - a_{t+1} - b_1 < \ell_3 - \sum_{j=t-1}^{t} a_j$; i.e.,

(j'_3): $\quad a_{t+1} > a_{t-1} + a_t - b_1.$

If $c = 4$, it suffices to prove that $\operatorname{Ext}^1_R(I_{D_2}, I_3)_0 = 0$ by the argument of (4.57). Indeed, (4.49) vanishes and we know that $\operatorname{depth}_{I(Z_2)} D_2 \geq 4$ implies the injection

$$\operatorname{Ext}^1_{D_3}(I_{D_3}/I^2_{D_3}, I_3)_0 \hookrightarrow \operatorname{Ext}^1_{D_2}(I_{D_2}/I^2_{D_2}, I_3)_0,$$

by (4.24), and that the latter Ext^1-group vanishes if $\operatorname{Ext}^1_R(I_{D_2}, I_3)_0 = 0$. Now exactly as in the first part of the proof of Corollary 4.2.43, we have $\operatorname{Ext}^1_R(I_{D_2}, I_3)_0 = 0$ provided

($\mathrm{j_4}$): $\quad a_{t+2} > a_{t-1} + a_t - b_1,$

and we conclude by Theorem 4.2.35. For the dimension formulas, we use Remark 4.2.32(2) and Corollary 4.2.34. □

Remark 4.2.45. Looking at the proofs of Corollaries 4.2.43 and 4.2.44, we get the following. Let $U \subset \overline{W(\underline{b};\underline{a})}$ be the subset where $\operatorname{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is smooth, and assume $a_0 \le a_1 \le \cdots \le a_{t+c-2}$ and $b_1 \le \cdots \le b_t$. Then U contains every X for which the flag

$$X = X_c \subset \cdots \subset X_i = \operatorname{Proj}(D_i) \subset \cdots \subset X_2 \subset \mathbb{P}^{n+c}$$

of Theorem 4.2.35 satisfies $\operatorname{depth}_{I(Z_i)} D_i \ge 4$ for $2 \le i \le c-2$, $\operatorname{depth}_{I(Z_{c-1})} D_{c-1} \ge 3$, and

(1) (j_3') if $c = 3$,
(2) (j_4) if $c = 4$,
(3) (j_5) to (j_c) if $c \ge 5$.

Moreover, if $3 \le c \le 4$, we can drop (j_3') and (j_4) provided we increase the depth assumption to $\operatorname{depth}_{I(Z_i)} D_i \ge 4$ for $2 \le i \le c-1$.

Corollaries 4.2.31 and 4.2.43 and Remark 4.2.32(1) can be improved a little bit if we increase $\operatorname{depth}_{I(Z_i)} D_i$. In fact, by Remark 1.2.22 we know that under the assumption $a_{i-\min(3,t)} \ge b_i$ for $\min(3,t) \le i \le t$, we can suppose $\operatorname{depth}_{I(Z_i)} D_i \ge 5$ for $i \ge 3$, letting $Z_i = \operatorname{Sing}(X_i)$. Since $Z_i \subset Z_{i-1}$, if we suppose

$$\operatorname{depth}_{I(Z_{i-2})} D_{i-2} \ge 5, \tag{4.58}$$

we get that

$$\operatorname{depth}_{I(Z_{i-2})} D_{i-1} \ge 4,$$

and hence $\operatorname{depth}_{I(Z_{i-1})} D_{i-1} \ge 4$, as well as $\operatorname{depth}_{I(Z_{i-2})} D_i \ge 3$. As in (4.19), we see that

$$\begin{aligned}\operatorname{Ext}^1_{D_{i-2}}(I_{i-2}, I_i) &\cong H^1_*(U_{i-2}, \mathcal{H}om_{\mathcal{O}_{X_{i-2}}}(\mathcal{I}_{i-2}, \mathcal{I}_i)) \\ &\cong H^1_*(U_{i-2}, \mathcal{H}om_{\mathcal{O}_{X_i}}(\mathcal{I}_{i-2} \otimes \mathcal{I}_i^*, \mathcal{O}_{X_i})) \\ &\cong H^1_*(U_{i-2}, \mathcal{O}_{X_i}(a_{t+i-3} - a_{t+i-2})) = 0.\end{aligned}$$

Arguing as in (4.23) and combining with (4.24), we get the injections

$$\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i) \hookrightarrow \operatorname{Ext}^1_{D_{i-1}}(I_{D_{i-1}}/I^2_{D_{i-1}}, I_i) \hookrightarrow \operatorname{Ext}^1_{D_{i-2}}(I_{D_{i-2}}/I^2_{D_{i-2}}, I_i). \tag{4.59}$$

In particular, if $\operatorname{Ext}^1_R(I_{D_{i-2}}, I_i)_0 = 0$ and if (4.58) hold, then

$$\operatorname{Ext}^1_{D_i}(I_{D_i}/I^2_{D_i}, I_i)_0 = 0.$$

Now looking at the proof of Corollary 4.2.43, we easily see that

$$\operatorname{Ext}^1_R(I_{D_{c-3}}, I_{c-1})_0 = 0$$

provided

$$a_{t+c-2} > \sum_{j=t-1}^{t+c-5} a_j - \sum_{j=0}^{c-6} a_j - b_1. \tag{4.60}$$

Hence if $(\mathrm{j_{c-1}})$ holds, then (4.60) holds because $a_{t+c-2} \geq a_{t+c-3}$, and it is superfluous to assume $(\mathrm{j_c})$ in Corollary 4.2.43. The argument requires $\operatorname{depth}_{I(Z_{c-3})} D_{c-3} \geq 5$, i.e., $c \geq 6$, and arguing slightly more general, we see that if $6 \leq i \leq c$, then $(\mathrm{j_i})$ is superfluous provided $(\mathrm{j_{i-1}})$ holds.

In particular, the conclusion of Corollary 4.2.43 holds if $(\mathrm{j_i})$ holds for any *odd* number i such that $5 \leq i \leq c$.

Remark 4.2.46. (1) In Corollary 4.2.43, the assumption $(\mathrm{j_i})$ is superfluous if $(\mathrm{j_{i-1}})$ holds, $6 \leq i \leq c$.

(2) Increasing $\operatorname{depth}_{I(Z_i)} D_i$ even more (say by assuming $a_{i-\min(4,t)} \geq b_i$ for $\min(4,t) \leq i \leq t$, cf. Remark 1.2.22), we can weaken $(\mathrm{j_k})$ (resp., $(\mathrm{i_k})$) conditions of Corollary 4.2.43 (resp., Corollary 4.2.31) further.

We would like to end this chapter with a nice conjecture which naturally arises in this context. Indeed, Theorem 4.2.7, Example 4.2.10 (a)–(d), Proposition 4.2.15, and Corollaries 4.2.23, 4.2.26, 4.2.30, 4.2.31, and 4.2.34 suggest – and prove in many cases – the following conjecture.

Conjecture 4.2.47. *Given integers* $a_0 \leq a_1 \leq \cdots \leq a_{t+c-2}$ *and* $b_1 \leq \cdots \leq b_t$*, we set* $\ell := \sum_{j=0}^{t+c-2} a_j - \sum_{i=1}^{t} b_i$ *and* $h_i := 2a_{t+1+i} + a_{t+2+i} + \cdots + a_{t+c-2} - \ell + n + c$ *for* $i = 0, 1, \ldots, c-3$*. Assume* $a_{i-\min([c/2]+1,t)} \geq b_i$ *for* $\min([c/2]+1, t) \leq i \leq t$*. Then we have*

$$\begin{aligned}
\dim W(\underline{b}; \underline{a}) = {} & \sum_{i,j} \binom{a_i - b_j + n + c}{n+c} - \sum_{i,j} \binom{a_i - a_j + n + c}{n+c} \\
& - \sum_{i,j} \binom{b_i - b_j + n + c}{n+c} + \sum_{j,i} \binom{b_j - a_i + n + c}{n+c} + \binom{h_0}{n+c} + 1 \\
& + \sum_{i=1}^{c-3} \left(\sum_{r+s=i} \sum_{\substack{0 \leq i_1 < \cdots < i_r \leq t+i \\ 1 \leq j_1 \leq \cdots \leq j_s \leq t}} (-1)^{i-r} \binom{h_i + a_{i_1} + \cdots + a_{i_r} + b_{j_1} + \cdots + b_{j_s}}{n+c} \right).
\end{aligned}$$

In particular, we would like to know if the above conjecture is at least true when the entries of $\mathcal{A}$ all have the same degree. More precisely, we have the following conjecture.

Conjecture 4.2.48. *Let* $W(\underline{0}; \underline{d})$ *be the locus of good determinantal schemes in* $\mathbb{P}^{n+c}$ *of codimension* c *given by the maximal minors of a* $t \times (t+c-1)$ *matrix with entries homogeneous forms of degree* d*. Then,*

$$\dim W(\underline{0}; \underline{d}) = t(t+c-1)\binom{d+n+c}{n+c} - t^2 - (t+c-1)^2 + 1.$$

This page intentionally left blank

Chapter 5

Determinantal Ideals, Symmetric Determinantal Ideals, and Open Problems

In this last chapter, we address for determinantal ideals $I \subset K[x_0, \ldots, x_n]$, i.e., ideals of codimension $c = (p-r+1)(q-r+1)$ generated by the $r \times r$ minors of a $p \times q$ homogeneous matrix $\mathcal{A}$ (see Definition 1.2.3), for symmetric determinantal ideals $I \subset K[x_0, \ldots, x_n]$, i.e., ideals of codimension $c = \binom{m-t+2}{2}$ generated by the $t \times t$ minors of an $m \times m$ homogeneous symmetric matrix $\mathcal{A}$ (see Definition 1.2.5), and for the three problems considered in the previous chapters for standard determinantal ideals. Namely, we address the following problems:

(1) CI-liaison class and G-liaison class of determinantal ideals,
(1′) CI-liaison class and G-liaison class of symmetric determinantal ideals,
(2) the multiplicity conjecture for determinantal ideals,
(2′) the multiplicity conjecture for symmetric determinantal ideals,
(3) unobstructedness and dimension of families of determinantal schemes,

and

(3′) unobstructedness and dimension of families of symmetric determinantal schemes.

We collect what is known, some open problems that naturally arise in this new context, and we add some conjectures raised in this work.

5.1 Liaison class of determinantal and symmetric determinantal ideals

In Section 2.3, we have generalized Gaeta's theorem and we have seen that standard determinantal schemes $X \subset \mathbb{P}^n$ are G-linked in a finite number of steps to a complete intersection; i.e., they are glicci (see Theorem 2.3.1). The following question should also be viewed as a generalization of Gaeta's theorem (see Theorem 2.1.6).

Question 5.1.1. *Is there only one G-liaison class containing ACM subschemes $X \subset \mathbb{P}^n$ of codimension c? Or, equivalently, are all ACM subschemes $X \subset \mathbb{P}^n$ glicci?*

Based on the results of Chapter 2, as well as those in [56], [13], [14], [15], [41], and [66], we would expect a yes answer to the last question. Notice that even in codimension 3, an affirmative answer to the above question will be a very interesting result. It will also be worthwhile to know if the following partial result is true.

Question 5.1.2. *Is any ACM curve $C_t \subset \mathbb{P}^4$ with a linear resolution,*

$$0 \longrightarrow R(-t-2)^{\frac{t^2+t}{2}} \longrightarrow R(-t-1)^{t^2+2t} \longrightarrow R(-t)^{\frac{t^2+3t+2}{2}} \longrightarrow I(C_t) \longrightarrow 0$$

glicci?

We know many examples of glicci, ACM curves $C_t \subset \mathbb{P}^4$ with a linear resolution. Indeed, any ACM curve $D_t \subset \mathbb{P}^4$ defined by the maximal minors of a $t \times (t+2)$ matrix with linear entries has a linear resolution and by Theorem 2.3.1 D_t is glicci. Nevertheless, not all ACM curves $C_t \subset \mathbb{P}^4$ with a linear resolution are standard determinantal. In fact, by Theorem 4.2.7, the family of such standard determinantal curves has dimension $\leq 3t^2+6t-3$. On the other hand, each component of the Hilbert scheme of curves of degree

$$d(D_t) = \binom{t+3}{4} - \binom{t+2}{4}$$

and genus

$$p_a(D_t) = (t-1)d(D_t) + 1 - \binom{t+3}{4}$$

has dimension $\geq \chi(\mathcal{N}_{D_t}) = 5d(D_t)+1-p_a(D_t)$. Thus, it is enough to take a value of t (for instance $t = 3, 4$) such that $3t^2+6t-3 \leq 5d(D_t)+1-p_a(D_t)$.

In the context of determinantal ideals, we would like to generalize the main result of Chapter 2 which states that standard determinantal ideals are glicci (see Theorem 2.3.1), and ask whether arbitrary determinantal ideals are glicci and whether symmetric determinantal ideals are glicci. More precisely, we consider determinantal schemes in $\mathbb{P}^n$, namely subschemes $X_{p,q,r} \subset \mathbb{P}^n$ of codimension

$(p-r+1)(q-r+1)$ defined by the $r \times r$, $r \leq \min(p,q)$, minors of a $p \times q$ homogeneous matrix $\mathcal{A}$. It is well known that $X_{p,q,r}$ is an ACM subscheme (see [20]) and hence, as a particular case of Question 5.1.1, we are led to pose the following question.

Question 5.1.3. *Is any determinantal subscheme $X_{p,q,r} \subset \mathbb{P}^n$ glicci or, equivalently, is any determinantal subscheme $X_{p,q,r} \subset \mathbb{P}^n$ G-linked in a finite number of steps to a complete intersection.*

This last question has been recently answered by E. Gorla. In [32], she has proved the following theorem.

Theorem 5.1.4. *Every determinantal subscheme $X_{p,q,r} \subset \mathbb{P}^n$ can be G-bilinked in r steps to a complete intersection. In particular, every determinantal subscheme $X_{p,q,r} \subset \mathbb{P}^n$ is glicci.*

Sketch of the proof. Let $\mathcal{A} = (F_{ij})_{1\leq j\leq q}^{1\leq i\leq p}$ be the homogeneous matrix associated with $X_{p,q,r} \subset \mathbb{P}^n$. So, $I(X_{p,q,r}) = I_r(\mathcal{A})$ and $X_{p,q,r} \subset \mathbb{P}^n$ has codimension $c = (p-r+1)(q-r+1)$. Let $I_r(\mathcal{L})$ be the ideal generated by the $r \times r$ minors of $\mathcal{L}$, where $\mathcal{L}$ is the subladder of $\mathcal{A}$ consisting of all the entries except for F_{pq} (after applying generic invertible row operations to $\mathcal{A}$). Let $\mathcal{N}$ be the submatrix obtained from $\mathcal{A}$ by deleting the last row and column, and let $X' \subset \mathbb{P}^n$ be the determinantal scheme with $I(X') = I_{r-1}(\mathcal{N})$.

By [32, Theorem 2.11], $I_r(\mathcal{L})$ is the saturated ideal of an ACM, generically complete intersection subscheme $Y \subset \mathbb{P}^n$ of codimension $c-1$ and $X_{p,q,r}, X' \subset Y$ are generalized divisors on Y. Moreover, $X_{p,q,r} \sim X' + aH$ for some $a > 0$, where $\sim$ denotes linear equivalence of generalized divisors on Y and H is a hyperplane divisor of Y. By Theorem 1.3.12, $X_{p,q,r}$ and X' are G-bilinked. Repeating this argument, we arrive after r steps to a complete intersection. Therefore, $X_{p,q,r}$ is glicci. $\square$

We consider symmetric determinantal subschemes in $\mathbb{P}^n$, i.e., subschemes $X^s_{m,t} \subset \mathbb{P}^n$ of codimension $\binom{m-t+2}{2}$ defined by the $t \times t$ minors of an $m \times m$ homogeneous symmetric matrix. We know that $X^s_{m,t}$ is an ACM subscheme (see [62, Theorem 1]) and we wonder:

Question 5.1.5. *Is $X^s_{m,t} \subset \mathbb{P}^n$ glicci? Or, equivalently, is any symmetric determinantal subscheme G-linked in a finite number of steps to a complete intersection?*

Example 5.1.6. Let V be the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, i.e., the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Its homogeneous ideal $I(V)$ is defined by the 2×2 minors of the generic symmetric matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix},$$

and it has the following minimal free R-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-4)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-3)^8 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-2)^6 \longrightarrow I(V) \longrightarrow 0.$$

Therefore, V is ACM and has degree 4 (notice that V is not a standard determinantal scheme). V is an ACM effective divisor on the rational normal scroll $S(0,1,2) \subset \mathbb{P}^5$ defined by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \end{pmatrix}.$$

Thus, V is glicci by [13, Theorem 4.10(i)].

In [60], we have generalized the above example to any codimension 3, ACM subscheme $X \subset \mathbb{P}^n$ defined by the submaximal minors of a $t \times t$ homogeneous symmetric matrix $\mathcal{A}$ and we have got the first important contribution to Question 5.1.5. To prove it we have to fix some notation.

Let $X \subset \mathbb{P}^n$ be a codimension 3, ACM scheme defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\dots,m}$, where $f_{ji} \in K[x_0, \dots, x_n]$ are homogeneous polynomials of degree $a_i + a_j$, and let $A = R/I(X)$ be the homogeneous coordinate ring of X. We denote by

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1 + a_2 & \dots & a_1 + a_m \\ a_1 + a_2 & 2a_2 & \dots & a_2 + a_m \\ \vdots & \vdots & & \vdots \\ a_1 + a_m & a_2 + a_m & \dots & 2a_m \end{pmatrix}$$

the degree matrix of $\mathcal{A}$. The determinant of $\mathcal{A}$ is a homogeneous polynomial of degree $\ell = 2(a_1 + a_2 + \cdots + a_m)$. Note that $a_i + a_j$ is a positive integer for all $1 \leq i \leq j \leq m$, while a_i does not necessarily need to be an integer. Let $\mathcal{B}$ be the matrix obtained by deleting the last row, let $I_B = I_{m-1}(\mathcal{B})$ be the ideal defined by the maximal minors of $\mathcal{B}$, and let $I_A = I_{m-1}(\mathcal{A})$ be the ideal generated by the submaximal minors of $\mathcal{A}$. Set $A = R/I_A = R/I(X)$ and $B = R/I_B$.

Remark 5.1.7. Assume $\mathrm{Char}(K) \neq 2$. After a basis change that preserves the symmetry of $\mathcal{A}$, if necessary, we have that the assumption $\mathrm{codim}_R A = 3$ implies that $\mathrm{codim}_R B = 2$ and I_B is Cohen–Macaulay. In fact, we know well that $\mathrm{codim}_R B \leq 2$. Following the approach of the proof of [8, Theorem 2], and strongly using that the matrix $\mathcal{A}$ is symmetric and $\mathrm{Char}(K) \neq 2$, we get (see also [31, Theorem 1.22] for the details)

$$ht(I_A/I_B) = ht(I_{t-1}(\mathcal{A})/I_{t-1}(\mathcal{B})) \leq 1.$$

Therefore, we obtain $ht(I_B) = ht(I_{t-1}(\mathcal{B})) \geq ht(I_A) = ht(I_{t-1}(\mathcal{A})) - 1 = 2$, and we are done.

Remark 5.1.8. With the above notations, let $Y \subset \mathbb{P}^n$ be the codimension 2, ACM scheme defined by the maximal minors of $\mathcal{B}$. We *claim* that Y is a generically complete intersection. In fact, consider

$$0 \longrightarrow F \xrightarrow{{}^t\mathcal{B}} G \longrightarrow I_B = I(Y) \longrightarrow 0$$

the resolution of $I(Y)$ given by Hilbert–Burch theorem. Let P be a minimal associated prime of $I(Y) = I_{t-1}(\mathcal{B})$. We have to see that $I(Y)_P$ is a complete intersection. We have $ht(P) = ht(I_{t-1}(\mathcal{B})) = 2 < 3 = ht(I(X)) = ht(I_{t-1}(\mathcal{A})) \leq ht(I_{t-2}(\mathcal{B}))$. So, $P \not\supseteq I_{t-2}(\mathcal{B})$. Denote by $\mu(I(Y)_P)$ the number of minimal generators of $I(Y)_P$. By [10, Proposition 16.3], $\mu(I(Y)_P) \leq 2$, and we are done.

Remark 5.1.9. If the entries of $\mathcal{A}$ and $\mathcal{B}$ are sufficiently general polynomials of degree a_i+a_j with a_i+a_j a positive integer for all $1 \leq i \leq j \leq m$, then $\operatorname{codim}_R B = 2$ and $\operatorname{codim}_R A = 3$. In fact, given rational numbers $a_i \in \mathbb{Q}$, $1 \leq i \leq m$, with $a_i + a_j$ a positive integer for all $1 \leq i \leq j \leq m$, we can consider the $m \times m$ homogeneous symmetric matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & x_1^{a_1+a_{m-1}} & x_0^{a_1+a_m} \\ 0 & 0 & \dots & 0 & 0 & x_1^{a_2+a_{m-2}} & x_0^{a_2+a_{m-1}} & x_2^{a_2+a_m} \\ 0 & 0 & \dots & 0 & x_1^{a_3+a_{m-3}} & x_0^{a_3+a_{m-2}} & x_2^{a_3+a_{m-1}} & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0^{a_1+a_m} & x_2^{a_2+a_m} & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and let $\mathcal{B}$ be the matrix obtained by deleting the last row of $\mathcal{A}$. We clearly have $\operatorname{codim}_R B = 2$ and $\operatorname{codim}_R A = 3$ which proves what we want.

Proposition 5.1.10. *With the above notation and assumptions,* $\operatorname{codim}_R A = 3$ *and* $\operatorname{codim}_R B = 2$*, we have an exact sequence,*

$$0 \longrightarrow (S^2 K_B)(2n+2+p) \longrightarrow B \longrightarrow A \longrightarrow 0,$$

where $p = 2a_m - 3\ell$ *and* $K_B = \operatorname{Ext}^2_B(B, R(-n-1))$ *is the canonical module of* B.

Proof. First of all, we observe that the degrees of the minors of $\mathcal{B}$ are $\ell - a_i - a_m$; i.e., I_B has the following minimal free R-resolution:

$$0 \longrightarrow \oplus_{i=1}^{m-1} R(-a_i + a_m - \ell) \xrightarrow{{}^t\mathcal{B}} \oplus_{i=1}^{m} R(a_i + a_m - \ell) \longrightarrow I_B \longrightarrow 0. \tag{5.1}$$

By [53, Theorem 3.1], I_A has a minimal free R-resolution of the following type (Józefiak's resolution):

$$\begin{aligned} 0 &\longrightarrow \oplus_{1\leq i<j\leq m} R(-a_i - a_j - \ell) \longrightarrow \oplus_{1\leq i,j\leq m} R(-\ell - a_i + a_j)/R(-\ell) \\ &\xrightarrow{\mathcal{A}} \oplus_{1\leq i\leq j\leq m} R(a_i + a_m - \ell) \longrightarrow I_A \longrightarrow 0. \end{aligned} \tag{5.2}$$

The natural injection $I_B \hookrightarrow I_A$ induces a map from the complex (5.1) to the complex (5.2) in the most obvious way; i.e., factors in the complex (5.1) which are not present in the complex (5.2) are mapped to zero, otherwise it is mapped by the identity.

Dualizing the exact sequence (5.1), we get

$$0 \longrightarrow R \longrightarrow \oplus_{i=1}^{m} R(-a_i - a_m + \ell) \xrightarrow{\mathcal{B}} \oplus_{i=1}^{m-1} R(a_i - a_m + \ell) \longrightarrow K_B(n+1) \longrightarrow 0$$

leading to the exact sequence (see [22, Theorem A2.10])

$$\begin{aligned} 0 \longrightarrow \oplus_{1 \le i < j \le m} R(-\ell - a_i - a_j) &\longrightarrow \oplus_{\substack{1 \le i \le m-1 \\ 1 \le j \le m}} R(-\ell + a_i - a_j) \\ \xrightarrow{\mathcal{B}} \oplus_{1 \le i < j \le m} R(-\ell + a_i + a_j) &\longrightarrow S^2 K_B(2n+2+p) \longrightarrow 0. \end{aligned} \tag{5.3}$$

One easily checks that the natural maps from the complex of I_A onto the complex of $S^2 K_B(2n+2+p)$ define a morphism of complexes, and hence we get the exact sequence

$$0 \longrightarrow I_B \longrightarrow I_A \longrightarrow (S^2 K_B)(2n+2+p) \longrightarrow 0,$$

and thus there is an exact sequence

$$0 \longrightarrow (S^2 K_B)(2n+2+p) \longrightarrow B \longrightarrow A \longrightarrow 0$$

which proves what we want. □

Proposition 5.1.11. *Assume* Char$(K) \neq 2$. *Let* $X \subset \mathbb{P}^n$ *be a codimension* 3, *ACM scheme defined by the submaximal minors of an* $m \times m$ *homogeneous symmetric matrix* $\mathcal{A}$. *Then* X *is glicci.*

Proof. Let Y be the codimension 2, ACM subscheme of $\mathbb{P}^n$ defined by the maximal minors of the $m \times (m-1)$ matrix $\mathcal{B}$ obtained by deleting the last row of $\mathcal{A}$, after a basis change that preserves the symmetry of $\mathcal{A}$, if necessary. By Remark 5.1.8, Y is a generically complete intersection. Hence Y satisfies the property G_0 and we have the concept of generalized divisors on Y.

By Proposition 5.1.10, $I(X)/I(Y) = (S^2 K_B)(2n+2+p)$, i.e., $X \in |-2K_Y - (2n+2+p)H|$, where K_Y is the canonical divisor on Y and H is a hyperplane divisor.

By Theorem 1.3.11, for $r \gg 0$, $G \sim rH - K_Y$ is AG and G-links X to an element of the linear system $|K_Y + \beta H|$ for a suitable $\beta \in \mathbb{Z}$. So, it suffices to check that an element of the linear system $|K_Y + \beta H|$ is glicci. To this end, we consider a codimension 3 standard determinantal scheme $D \subset Y \subset \mathbb{P}^n$ defined by the maximal minors of the matrix $[\mathcal{B}, \mathcal{L}]$ obtained by adding to $\mathcal{B}$ a sufficiently general column $\mathcal{L}$ such that $[\mathcal{B}, \mathcal{L}]$ is again homogeneous. By [56, Theorem 3.6], $D \in |K_Y + tH|$ for some $t \in \mathbb{Z}$ and D is glicci. Moreover, by [56, Corollary 5.13], $K_Y + tH$ and D are G-bilinked, so any effective divisor of type $K_Y + dH$ is ACM and glicci, and we are done. □

The above proposition has been recently generalized by E. Gorla [31]. In fact, she answered affirmatively Question 5.1.5 and she proved thefollowing theorem.

Theorem 5.1.12. *Any codimension* $\binom{m-t+2}{2}$ *symmetric determinantal subscheme* $X^s_{m,t}$ *of* $\mathbb{P}^n$ *defined by the* $t\times t$ *minors of an* $m\times m$ *symmetric homogeneous matrix* $\mathcal{A}$ *is glicci.*

Proof. See [31, Corollary 2.7]. □

5.2 The multiplicity conjecture for determinantal and symmetric determinantal ideals

Let us recall the Herzog–Huneke–Srinivasan conjecture (multiplicity conjecture) about the multiplicity of perfect ideals.

Conjecture 5.2.1. *Let* $I \subset R$ *be a graded Cohen–Macaulay ideal of codimension* c. *Consider the minimal graded free* R*-resolution of* R/I,

$$0 \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

and set $m_i(I) := \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$ *and* $M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$. *Then, we have*

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}.$$

The main result of Chapter 3 states that the above conjecture works for standard determinantal ideals (see Theorem 3.2.6), and we would like to know whether the multiplicity conjecture is true for arbitrary determinantal ideals and for symmetric determinantal ideals. So, we are led to pose the following questions.

Question 5.2.2. *Let* $I_{p,q,r} \subset K[x_1,\dots,x_n]$ *be a determinantal ideal generated by the* $r\times r$, $r \leq \min(p,q)$, *minors of a* $p\times q$, *homogeneous matrix* $\mathcal{A}$. *Does* $I_{p,q,r}$ *satisfy Conjecture* 5.2.1*?*

Question 5.2.3. *Let* $I^s_{m,t} \subset K[x_1,\dots,x_n]$ *be a symmetric determinantal ideal generated by the* $t\times t$ *minors of an* $m\times m$, $m \geq t$, *homogeneous symmetric matrix* $\mathcal{A}$. *Does* $I^s_{m,t}$ *satisfy Conjecture* 5.2.1*?*

The first contribution to Question 5.2.3 was given in [69], where we prove the multiplicity conjecture of Herzog, Huneke, and Srinivasan for K-algebras $K[x_1,\dots,x_n]/I$, where I is a symmetric determinantal ideal generated by the submaximal minors of an $m\times m$ homogeneous symmetric matrix. A classical homogeneous ideal that can be generated by the submaximal minors of an $m\times m$ homogeneous symmetric matrix is the ideal of the Veronese surface $X \subset \mathbb{P}^5$ (cf. Example 5.1.6).

Let $I \subset R = K[x_1, \dots, x_n]$ be a codimension 3, symmetric determinantal ideal generated by the submaximal minors of an $m \times m$ homogeneous symmetric matrix,

$$\mathcal{A} = (f_{ji})_{i,j=1,\dots,m},$$

where $f_{ji} \in K[x_1, \dots, x_n]$ are homogeneous polynomials of degree $a_i + a_j$, i.e., $I = I_{m-1}(\mathcal{A})$. As usual we denote by

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1 + a_2 & \dots & a_1 + a_m \\ a_1 + a_2 & 2a_2 & \dots & a_2 + a_m \\ \vdots & \vdots & & \vdots \\ a_1 + a_m & a_2 + a_m & \dots & 2a_m \end{pmatrix}$$

the degree matrix of $\mathcal{A}$. We may assume, without loss of generality, that $a_1 \leq a_2 \leq \cdots \leq a_m$. The determinant of $\mathcal{A}$ is a homogeneous polynomial of degree $\ell := 2(a_1 + a_2 + \cdots + a_m)$. Moreover, the degree matrix $\mathcal{U}$ is completely determined by $a_1, a_2, \dots, a_m$ and the graded Betti numbers in the minimal free R-resolution of $R/I_{m-1}(\mathcal{A})$ depend only upon the integers $a_1, a_2, \dots, a_m$ as we will describe now. To this end, we recall Józefiak's result about the resolution of ideals generated by submaximal minors of a symmetric matrix.

Let S be a commutative ring with identity and let $X = (x_{ij})$ be an $m \times m$ symmetric matrix with entries in S. Write $Y = (y_{ij})$ for the matrix of cofactors of X, i.e., $y_{ij} = (-1)^{i+1} X_j^i$ where X_j^i stands for the minor of X obtained by deleting the ith column and the jth row of X. The matrix Y is also a symmetric matrix. Let $M_m(S)$ be the free S-module of all $m \times m$ matrices over S and let $A_m(S)$ be the free S-submodule of $M_m(S)$ consisting of all alternating matrices. Denote by $tr : M_m(S) \longrightarrow S$ the trace map. In [53, Theorem 3.1], T. Józefiak proved that the free complex of length 3 associated with X,

$$0 \longrightarrow A_m(S) \xrightarrow{d_3} Ker(M_m(S) \xrightarrow{tr} S) \xrightarrow{d_2} M_m(S)/A_m(S) \xrightarrow{d_1} S,$$

where the corresponding differentials are defined as follows:

$d_3(A) = AX$,
$d_2(N) = XN \bmod A_m(S)$, and
$d_1(M \bmod A_m(S)) = tr(YM)$,

is acyclic and gives a free resolution of $S/I_{t-1}(X)$. So, we obtain the following proposition.

Proposition 5.2.4. *Let $I \subset R = K[x_1, \dots, x_n]$ be a symmetric determinantal ideal of codimension 3 generated by the submaximal minors of an $m \times m$ symmetric matrix $\mathcal{A}$. Let*

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1 + a_2 & \dots & a_1 + a_m \\ a_1 + a_2 & 2a_2 & \dots & a_2 + a_m \\ \vdots & \vdots & & \vdots \\ a_1 + a_m & a_2 + a_m & \dots & 2a_m \end{pmatrix}$$

be the degree matrix and $\ell := 2(a_1 + a_2 + \cdots + a_m)$. *Then, we have*

(1) $m_1(I) = \ell - 2a_m$ *and* $M_1(I) = \ell - 2a_1$,
(2) $m_2(I) = \ell - a_m + a_1$ *and* $M_2(I) = \ell - a_1 + a_m$,
(3) $m_3(I) = \ell + a_1 + a_2$ *and* $M_3(I) = \ell + a_{t-1} + a_m$, *and*
(4) $\beta_1(R/I) = \binom{m+1}{2}$, $\beta_2(R/I) = m^2 - 1$, *and* $\beta_3(R/I) = \binom{m}{2}$.

Proof. By [53, Theorem 3.1], I has a minimal free R-resolution of the following type:

$$0 \longrightarrow \oplus_{1\le i<j\le m} R(-a_i - a_j - \ell) \longrightarrow (\oplus_{1\le i,j\le m} R(-\ell - a_i + a_j))/R(-\ell)$$
$$\longrightarrow \oplus_{1\le i\le j\le m} R(a_i + a_m - \ell) \longrightarrow I \longrightarrow 0.$$

So, the maximum and minimum degree shifts at the ith step are given by

$$m_1(I) = \min_{1\le i\le j\le m} \{a_i + a_m - \ell\} = \ell - 2a_m,$$
$$M_1(I) = \max_{1\le i\le j\le m} \{a_i + a_m - \ell\} = \ell - 2a_1;$$

$$m_2(I) = \min_{1\le i,j\le m} \{\ell + a_i - a_j))\} = \ell - a_m + a_1,$$
$$M_2(I) = \max_{1\le i,j\le m} \{\ell + a_i - a_j))\}(I) = \ell - a_1 + a_m;$$

$$m_3(I) = \min_{1\le i<j\le m} \{a_i + a_j + \ell\} = \ell + a_1 + a_2,$$
$$M_3(I) = \max_{1\le i<j\le m} \{a_i + a_j + \ell\} = \ell + a_{m-1} + a_m.$$

Moreover, the i-the total Betti numbers are given by,

$$\beta_1(R/I) = \binom{m+1}{2}, \quad \beta_2(R/I) = m^2 - 1 \quad \text{and} \quad \beta_3(R/I) = \binom{m}{2},$$

which proves what we want. □

We will now compute the multiplicity of a symmetric determinantal ideal in terms of the corresponding degree matrix.

Lemma 5.2.5. *Let* $I \subset R$ *be symmetric determinantal ideal of codimension* 3 *generated by the submaximal minors of a homogeneous symmetric matrix* $\mathcal{A}$. *Let*

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1 + a_2 & \dots & a_1 + a_m \\ a_1 + a_2 & 2a_2 & \dots & a_2 + a_m \\ \vdots & \vdots & & \vdots \\ a_1 + a_m & a_2 + a_m & \dots & 2a_m \end{pmatrix}$$

be the degree matrix and $\ell := 2(a_1 + a_2 + \cdots + a_m)$*. Then, we have,*

$$e(R/I) = \frac{\ell^3 - 8\sum_{i=1}^{m} a_i^3}{6}.$$

Proof. By [53, Theorem 3.1], I has a minimal free R-resolution of the following type:

$$0 \longrightarrow \oplus_{1\le i<j\le m} R(-a_i - a_j - \ell) \longrightarrow (\oplus_{1\le i,j\le m} R(-\ell - a_i + a_j))/R(-\ell) \\ \longrightarrow \oplus_{1\le i\le j\le m} R(a_i + a_m - \ell) \longrightarrow I \longrightarrow 0.$$

To simplify the computation we denote it as

$$0 \longrightarrow \oplus_{i=1}^{r} R(-\gamma_i) \longrightarrow \oplus_{j=1}^{s} R(-\beta_j) \longrightarrow \oplus_{k=1}^{p} R(-\alpha_k) \\ \longrightarrow R \longrightarrow R/I \longrightarrow 0. \tag{5.4}$$

A straightforward computation, taking into account the precise values of α_k, β_j, and γ_i ($\alpha_k = \ell - a_t - a_s$, $\beta_j = a_t - a_s + \ell$ and $\gamma_i = a_t + a_s + \ell$), gives us

$$p + r = s + 1, \tag{5.5}$$

$$\sum_{k=1}^{p} \alpha_k + \sum_{i=1}^{r} \gamma_i = \sum_{j=1}^{s} \beta_j, \tag{5.6}$$

$$\sum_{k=1}^{p} \alpha_k^2 + \sum_{i=1}^{r} \gamma_i^2 = \sum_{j=1}^{s} \beta_j^2. \tag{5.7}$$

We will now prove that

$$e(R/I) = \frac{1}{6}\left(\sum_{k=1}^{p} \alpha_k^3 - \sum_{j=1}^{s} \beta_j^3 + \sum_{i=1}^{r} \gamma_i^3\right). \tag{5.8}$$

To this end, we may also assume that R/I is a 1-dimensional ring, and hence we have

$$e(R/I) = \dim_K (R/I)_v \quad \text{for} \quad v \gg 0.$$

Therefore, using the exact sequence (5.4) together with equalities (5.5)–(5.7), we obtain

$$\begin{aligned} e(R/I) = \dim_K R_v - &\sum_{k=1}^{p} \dim_K R(-\alpha_k)_v \\ &+ \sum_{j=1}^{s} \dim_K R(-\beta_j)_v - \sum_{i=1}^{r} \dim_K R(-\gamma_i)_v \\ = \binom{v+3}{3} - &\sum_{k=1}^{p} \binom{v-\alpha_k+3}{3} + \sum_{j=1}^{s} \binom{v-\beta_j+3}{3} - \sum_{i=1}^{r} \binom{v-\gamma_i+3}{3} \end{aligned}$$

$$= \frac{v^3+6v^2+11v+6}{6} - \sum_{k=1}^{p} \frac{(v-\alpha_k)^3+6(v-\alpha_k)^2+11(v-\alpha_k)+6}{6}$$
$$+ \sum_{j=1}^{s} \frac{(v-\beta_j)^3+6(v-\beta_j)^2+11(v-\beta_j)+6}{6}$$
$$- \sum_{i=1}^{r} \frac{(v-\gamma_i)^3+6(v-\gamma_i)^2+11(v-\gamma_i)+6}{6}$$
$$= \frac{1}{6}\left(\sum_{k=1}^{p}\alpha_k^3 - \sum_{j=1}^{s}\beta_j^3 + \sum_{i=1}^{r}\gamma_i^3\right).$$

Finally, substituting α_k, β_j, and γ_i by their values given in (5.2), we obtain

$$e(R/I) = \frac{1}{6}\left(\sum_{k=1}^{p}\alpha_k^3 - \sum_{j=1}^{s}\beta_j^3 + \sum_{i=1}^{r}\gamma_i^3\right)$$
$$= \frac{1}{6}\left(\sum_{1\le i\le j\le m}(\ell-a_i-a_j)^3 - \left((m-1)\ell^3 + \sum_{\substack{1\le i,j\le m\\ i\ne j}}(\ell+a_i-a_j)^3\right)\right)$$
$$+ \frac{1}{6}\sum_{1\le i<j\le m}(\ell+a_i+a_j)^3$$
$$= \frac{1}{6}(\ell^3 - 3\ell^2(2a_1+2a_2+\cdots+2a_m) - (8a_1^3+8a_2^3+\cdots+8a_m^3))$$
$$+ \frac{1}{6}\left(3\ell\left(4a_1^2+4a_2^2+\cdots+4a_m^2+8\sum_{1\le i<j\le m}a_ia_j\right)\right)$$
$$= \frac{1}{6}(\ell^3 - 3\ell^2(\ell) + 3\ell(\ell^2) - (8a_1^3+8a_2^3+\cdots+8a_m^3))$$
$$= \frac{\ell^3 - 8\sum_{i=1}^{m}a_i^3}{6}. \qquad \square$$

We are now ready to prove that Herzog–Huneke–Srinivasan conjecture is true for codimension 3, perfect ideals generated by the submaximal minors of a homogeneous symmetric matrix (see [69]).

Theorem 5.2.6. *Let $I \subset R$ be a symmetric determinantal ideal of codimension 3 generated by the submaximal minors of an $m\times m$ homogeneous symmetric matrix. Then the following lower and upper bounds hold:*

(1) *$e(R/I) \ge m_1m_2m_3/6$, and*
(2) *$e(R/I) \le M_1M_2M_3/6$.*

Moreover, the bounds are reached if and only if R/I has a pure resolution.

Proof. (1) Since $e(R/I) = \frac{\ell^3 - 8\sum_{i=1}^m a_i^3}{6}$ (Lemma 5.2.5), we only need to see that $m_1(I)m_2(I)m_3(I) \leq \ell^3 - 8\sum_{i=1}^m a_i^3$. Using the hypothesis $a_1 \leq a_2 \leq \cdots \leq a_m$ and applying Proposition 5.2.4, we get

$$\begin{aligned} m_1(I)m_2(I)m_3(I) &= (\ell - 2a_m)(\ell - a_m + a_1)(\ell + a_1 + a_2) \\ &\leq (\ell - 2a_m)\ell(\ell + 2a_m) \\ &= (\ell^2 - 4a_m^2)\ell = \ell^3 - 4a_m^2\ell \\ &= \ell^3 - 4a_m^2(2a_1 + 2a_2 + \cdots + 2a_m) \\ &= \ell^3 - 8a_1a_m^2 - 8a_2a_m^2 - \cdots - 8a_{m-1}a_m^2 - 8a_m^3 \\ &\leq \ell^3 - 8a_1^3 - 8a_2^3 - \cdots - 8a_m^3, \end{aligned}$$

which proves (1).

(2) Let us now check that $M_1(I)M_2(I)M_3(I) \geq \ell^3 - 8\sum_{i=1}^m a_i^3$. Using again the hypothesis $a_1 \leq a_2 \cdots \leq a_m$ and applying Proposition 5.2.4, we get

$$\begin{aligned} M_1(I)M_2(I)M_3(I) &= (\ell - 2a_1)(\ell - a_1 + a_m)(\ell + a_{m-1} + a_m) \\ &\geq (\ell - 2a_1)\ell(\ell + 2a_1) \\ &= (\ell^2 - 4a_1^2)\ell = \ell^3 - 4a_1^2\ell \\ &= \ell^3 - 4a_1^2(2a_1 + 2a_2 + \cdots + 2a_m) \\ &= \ell^3 - 8a_1^3 - 8a_2a_1^2 - \cdots - 8a_{m-1}a_1^2 - 8a_ta_1^2 \\ &\geq \ell^3 - 8a_1^3 - 8a_2^3 - \cdots - 8a_m^3, \end{aligned}$$

which proves (2).

Moreover, we observe that the inequalities are equalities if and only if $a_1 = a_2 = \cdots = a_m$ if and only if R/I has a pure resolution. ☐

In the next example we illustrate that the bounds given in Theorem 5.2.6 are optimal.

Example 5.2.7. Let $I \subset R$ be a codimension 3 symmetric determinantal ideal generated by the submaximal minors of an $m \times m$ homogeneous symmetric matrix whose all entries are homogeneous polynomials of fixed degree $1 \leq d \in \mathbb{Z}$. I has a minimal graded free R-resolution of the following type:

$$\begin{aligned} 0 \longrightarrow S(-d(m+1))^{\binom{m}{2}} \longrightarrow S(-dm)^{m^2-1} \\ \longrightarrow S(-d(m-1))^{\binom{m+1}{2}} \longrightarrow S \longrightarrow S/I \longrightarrow 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} m_1(I) = M_1(I) &= (m-1)d, \\ m_2(I) = M_2(I) &= md, \\ m_3(I) = M_3(I) &= (m+1)d. \end{aligned}$$

Therefore, we conclude that

$$e(S/I) = \frac{\prod_{i=1}^{3} m_i(I)}{3!} = \frac{\prod_{i=1}^{3} M_i(I)}{3!} = \frac{d^3 m(m^2-1)}{6}.$$

To end this section, we will bound the ith total Betti number of codimension 3 perfect ideals generated by the submaximal minors of a homogeneous symmetric matrix in terms of the shifts in its minimal free R-resolution.

Remark 5.2.8. Let $I \subset R$ be a perfect ideal of codimension 3 generated by the submaximal minors of a homogeneous symmetric matrix. It is worthwhile to point out that the ith total Betti number $\beta_i(R/I)$ in the minimal free R-resolution of R/I depend only upon the size $t \times t$ of the homogeneous symmetric matrix $\mathcal{A}$ associated with I.

Theorem 5.2.9. *Let $I \subset R$ be a perfect ideal of codimension 3 generated by the submaximal minors of a $t \times t$ homogeneous symmetric matrix. Then, we have*

$$\prod_{1\le j<i} \frac{m_j}{M_i - m_j} \prod_{i<j\le 3} \frac{m_j}{M_j - m_i} \le \beta_i(R/I) \le \frac{1}{(i-1)!\cdot(c-i)!} \prod_{j\ne i} M_j \qquad (5.9)$$

for $1 \le i \le 3$. In addition, the bounds are reached for all i if and only if R/I has a pure resolution if and only if $a_1 = \cdots = a_t$.

Proof. We will first prove the result for J, where $J \subset R$ is a codimension 3 perfect ideal generated by the submaximal minors of a $t\times t$ homogeneous symmetric matrix $\mathcal{A}$ with linear entries. In this case, for all $1 \le i \le 3$, we have (see Proposition 5.2.4)

$$m_i(J) = M_i(J) = t + i - 2,$$
$$\beta_1(R/J) = \binom{t+1}{2}, \quad \beta_2(R/J) = t^2 - 1, \quad \beta_3(R/J) = \binom{t}{2}.$$

Therefore, R/J has a pure resolution and it follows from [45] and [51] that

$$\begin{aligned}
&\prod_{1\le j<i} \frac{m_j(J)}{M_i(J) - m_j(J)} \prod_{i<j\le 3} \frac{m_j(J)}{M_j(J) - m_i(J)} \\
&\quad = \prod_{1\le j<i} \frac{t+j-2}{i-j} \prod_{i<j\le 3} \frac{t+j-2}{j-i} \\
&\quad = \beta_i(R/J) \\
&\quad = \prod_{1\le j<i} \frac{t+j-2}{i-j} \prod_{i<j\le 3} \frac{t+j-2}{j-i}
\end{aligned}$$

$$= \prod_{1\le j<i} \frac{M_j(J)}{m_i(J)-M_j(J)} \prod_{i<j\le 3} \frac{M_j(J)}{m_j(J)-M_i(J)}$$
$$= \frac{1}{(i-1)!\cdot(3-i)!} \prod_{j\neq i} M_j(J).$$

We will now prove the general case. Let I be a perfect ideal of codimension 3 generated by the submaximal minors of a $t\times t$ homogeneous symmetric matrix, let

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1+a_2 & \dots & a_1+a_t \\ a_1+a_2 & 2a_2 & \dots & a_2+a_t \\ \vdots & \vdots & & \vdots \\ a_1+a_t & a_2+a_t & \dots & 2a_t \end{pmatrix}$$

be its degree matrix and $\ell := 2(a_1+a_2+\dots+a_t)$. Since, for all $1\le i\le 3$, we have

$$M_i(I) \ge m_i(I) \ge t+i-2 = m_i(J) = M_i(J),$$

it follows from Proposition 5.2.4(3) and Remark 5.2.8 that

$$\begin{aligned}\beta_i(R/I) = \beta_i(R/J)& \\ &= \prod_{1\le j<i} \frac{t+j-2}{i-j} \prod_{i<j\le 3} \frac{t+j-2}{j-i} \\ &= \frac{1}{(i-1)!\cdot(3-i)!} \prod_{j\neq i} M_j(J) \\ &\le \frac{1}{(i-1)!\cdot(3-i)!} \prod_{j\neq i} M_j(I),\end{aligned}$$

and this completes the proof of the upper bound.

Let us now prove the lower bound. Using again Proposition 5.2.4 and Remark 5.2.8, we have

$$\begin{aligned}\frac{m_1}{M_3-m_1}\cdot\frac{m_2}{M_3-m_2} &= \frac{\ell-2a_t}{3a_t+a_{t-1}}\cdot\frac{\ell-a_t+a_1}{a_{t-1}+2a_t-a_1} \\ &\le \frac{\ell-2a_t}{4a_{t-1}}\cdot\frac{\ell-a_t+a_1}{2a_t} \\ &\le \frac{2(t-1)a_{t-1}}{4a_{t-1}}\cdot\frac{2ta_t}{2a_t} \\ &= \frac{(t-1)t}{2} = \beta_3(R/J) = \beta_3(R/I).\end{aligned}$$

Analogously, we obtain

$$\frac{m_1}{M_2 - m_1} \cdot \frac{m_3}{M_3 - m_2} \leq (t-1)(t+1) = \beta_2(R/J) = \beta_2(R/I)$$

and

$$\frac{m_2}{M_2 - m_1} \cdot \frac{m_3}{M_3 - m_1} \leq \frac{t(t+1)}{2} = \beta_1(R/J) = \beta_1(R/I),$$

and this completes the proof of the lower bound. It is easy to see that the inequality is an equality for all i if and only if $a_1 = a_2 \cdots = a_t$ if and only if R/I has a pure resolution. $\square$

We would like to end this section with a nice conjecture which naturally arise in this context. Indeed, Theorems 3.2.9 and 5.2.9 together with the main results in [71] and [78] suggest – and prove in many cases – the following conjecture.

Conjecture 5.2.10. *Let $I \subset R$ be a perfect ideal of codimension c. Then, we have*

$$\prod_{1 \leq j < i} \frac{m_j}{M_i - m_j} \prod_{i < j \leq c} \frac{m_j}{M_j - m_i} \leq \beta_i(R/I) \leq \frac{1}{(i-1)! \cdot (c-i)!} \prod_{j \neq i} M_j \qquad (5.10)$$

for $1 \leq i \leq c$. In addition, the bounds are reached for all i if and only if R/I has a pure resolution.

5.3 Unobstructedness and dimension of families of determinantal and symmetric determinantal ideals

The goal of this section is to write down a lower bound for the dimension of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ at $[X]$, where $X \subset \mathbb{P}^n$ is a codimension 3, ACM scheme defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix. We will also analyze when the mentioned bound is sharp. A classical scheme that can be constructed in this way is the Veronese surface $X \subset \mathbb{P}^5$. Indeed, the Veronese surface $X \subset \mathbb{P}^5$ can be defined by the 2×2 minors of the homogeneous symmetric matrix (see Example 5.1.6)

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

We keep the notation introduced in Sections 5.1 and 5.2. So, $X \subset \mathbb{P}^n$ will be a codimension 3, ACM scheme defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\ldots,m}$, where $f_{ji} \in K[x_0, \ldots, x_n]$ are homogeneous polynomials of degree $a_i + a_j$, and let $A = R/I(X)$ be the homogeneous coordinate ring of X. We denote by

$$\mathcal{U} = \begin{pmatrix} 2a_1 & a_1+a_2 & \dots & a_1+a_m \\ a_1+a_2 & 2a_2 & \dots & a_2+a_m \\ \vdots & \vdots & & \vdots \\ a_1+a_m & a_2+a_m & \dots & 2a_m \end{pmatrix}$$

the degree matrix of $\mathcal{A}$ and we set $\ell := 2(a_1 + a_2 + \cdots + a_m)$. Let $\mathcal{N}$ be the matrix obtained by deleting the last row, let $I_B = I_{m-1}(\mathcal{N})$ be the ideal defined by the maximal minors of $\mathcal{N}$, and let $I_A = I_{m-1}(\mathcal{A})$ be the ideal generated by the submaximal minors of $\mathcal{A}$. Set $A = R/I_A = R/I(X)$ and $B = R/I_B$. By Remark 5.1.7, after a basis change that preserves the symmetry of $\mathcal{A}$, if necessary, we have $\operatorname{codim}_R B = 2$.

To use deformation theory related to the flag of surjections

$$R \longrightarrow B \longrightarrow A \cong B/I_{A/B},$$

it is necessary to compute the groups

$$\begin{aligned} {}_0\operatorname{Ext}^1_B(I_B/I_B^2, I_{A/B}) \quad & (\text{or } {}_0\operatorname{Ext}^1_R(I_B, I_{A/B})); \\ {}_0\operatorname{Ext}^1_A(I_{A/B}/I_{A/B}^2, A) \quad & (\text{or } {}_0\operatorname{Ext}^1_B(I_{A/B}, A)). \end{aligned}$$

Lemma 5.3.1. *With the above notation and assumptions,*

$$\operatorname{codim}_R A = 3 \qquad \text{and} \qquad \operatorname{codim}_R B = 2,$$

suppose, in addition, $a_1 \le a_2 \le \cdots \le a_m$. Then

$${}_0\operatorname{Ext}^1_B(I_B/I_B^2, S^2K_B)(2n+2+p) = {}_0\operatorname{Ext}^1_R(I_B, (S^2K_B)(2n+2+p)) = 0$$

provided $a_m > 3a_{m-1}$ and

$${}_0\operatorname{Hom}_R(I_B, S^2K_B)(2n+2+p) = 0$$

provided $a_m > 2a_{m-1} - a_1$.

Proof. We apply the functor ${}_0\operatorname{Hom}_R(-, (S^2K_B)(2n+2+p))$ to the exact sequence (5.1) and we use the exact sequence (5.3) to deduce that ${}_0\operatorname{Ext}^1_R(I_B, S^2K_B)(2n+2+p) = 0$ provided we can show that

$${}_p\operatorname{Hom}(\oplus_{i=1}^{m-1} R(-a_i + a_m - \ell), \oplus_{1\le i\le j\le m-1} R(a_i + a_j - 2a_m + 2\ell)) = 0;$$

i.e., provided $R(a_i + a_j + a_k - 3a_m + 3\ell)_p = 0$ for all $i, j, k \le m-1$ or, equivalently, $a_m > 3a_{m-1}$.

To see the vanishing of the ${}_0$Hom-group, it suffices to show that

$${}_p\operatorname{Hom}(\oplus_{i=1}^{m} R(a_i + a_m - \ell), \oplus_{1\le i\le j\le m-1} R(a_i + a_j - 2a_m + 2\ell)) = 0,$$

and this last ${}_0$Hom-group vanishes provided $2a_{m-1} - a_1 - a_m > 0$. $\square$

Given rational numbers $a_1, \ldots, a_m$ such that $a_i + a_j \in \mathbb{Z}$ for all i, j, we denote by $S(\underline{a}) = S(a_1, \ldots, a_m)$ the irreducible family of codimension 3 symmetric determinantal subschemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\ldots,m}$, where $f_{ji} \in K[x_0, \ldots, x_n]$ is a homogeneous polynomial of degree $a_j + a_i$. Our next goal is to determine a lower bound for the dimension of the irreducible component $\mathcal{S}(\underline{a})$ of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ containing $S(\underline{a})$. To this end, set

$$\begin{aligned} a &:= {}_0\hom_B(I_B/I_B^2, I_{A/B}) - {}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) + {}_0\operatorname{ext}^2_B(I_B/I_B^2, I_{A/B}),\\ b &:= {}_0\hom_B(I_{A/B}, B) - {}_0\operatorname{ext}^1_B(I_{A/B}, B),\\ e &:= {}_0\operatorname{ext}^2_B(I_{A/B}, I_{A/B}). \end{aligned}$$

Proposition 5.3.2. *Set* $h^i_A := \dim_0 H^i(R, A, A)$*. If* $\operatorname{codim}_R B = 2$*, then*

$$b - 1 - e + \dim(N_B)_0 - a \leq h^1_A - h^2_A \leq \dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^n).$$

Proof. We consider the diagram

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & {}_0\operatorname{Hom}(I_B, I_{A/B}) & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & H^0(\widetilde{N_B}) & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
{}_0\operatorname{Hom}(I_{A/B}, A) & \hookrightarrow & {}_0H^1 & \rightarrow & {}_0\operatorname{Hom}_R(I_B, A) & \rightarrow & {}_0H^2(B, A, A) & \rightarrow & {}_0H^2 & \xrightarrow{\psi} & {}_0H^2(R, B, A) & \rightarrow \mathcal{C} \rightarrow & 0\\
 & & & & \downarrow & & & & & & \| & & \\
 & & & & {}_0\operatorname{Ext}^1(I_B/I_B^2, I_{A/B}) & & & & & & {}_0\operatorname{Ext}^1(I_B/I_B^2, A) & & \\
 & & & & \downarrow & & & & & & \| & & \\
 & & & & 0 & & & & & & {}_0\operatorname{Ext}^2(I_B/I_B^2, I_{A/B}) & &
\end{array}$$

where $H^i = H^i(R, A, A)$ and $C = \operatorname{coker}(\psi)$. Note that

$${}_0\operatorname{Ext}^i(I_B/I_B^2, B) \cong {}_0\operatorname{Ext}^i(I_B/I_B^2 \otimes K_B, K_B) = 0 \quad \text{for } i \leq 2$$

because B is licci, $\widetilde{K_B|U}$ is locally free, and $\operatorname{depth}_{I(Z)} B \geq 2$. Therefore, it suffices to show that

$${}_0\hom_B(I_{A/B}, A) - {}_0h^2(B, A, A) + \dim C \geq b - 1 - e.$$

Since one knows that ${}_0H^2(B, A, A) \hookrightarrow {}_0\operatorname{Ext}^1_B(I_{A/B}, A)$, it is enough to see that

$${}_0\hom_B(I_{A/B}, A) - {}_0\operatorname{ext}^1_B(I_{A/B}, A) \geq b - 1 - e.$$

To this end, we apply ${}_0\operatorname{Hom}_B(I_{A/B}, -)$ to the exact sequence

$$0 \longrightarrow I_{A/B} \longrightarrow B \longrightarrow A \longrightarrow 0,$$

and we get

$$\begin{aligned} {}_0\operatorname{Hom}_B(I_{A/B}, I_{A/B}) \longrightarrow {}_0\operatorname{Hom}_B(I_{A/B}, B) \longrightarrow {}_0\operatorname{Hom}_B(I_{A/B}, A)\\ \longrightarrow {}_0\operatorname{Ext}^1_B(I_{A/B}, I_{A/B}) \longrightarrow {}_0\operatorname{Ext}^1_B(I_{A/B}, B). \end{aligned} \tag{5.11}$$

Claim. ${}_0\hom_B(I_{A/B}, I_{A/B}) = 1$ and ${}_0\operatorname{Ext}^1_B(I_{A/B}, I_{A/B}) = 0$.

Proof of the claim. Since, by Proposition 5.1.10, $I_{A/B} = S^2K_B(2n+2+p)$, $I_{A/B}$ is a maximal Cohen–Macaulay module and we have $\operatorname{depth}_{I(Z)} I_{A/B} \geq 3$ provided we assume $\operatorname{depth}_{I(Z)} B = 3$. Hence,

$$\begin{aligned}\operatorname{Ext}^1(I_{A/B}, I_{A/B}) &\cong \operatorname{Ext}^1_{\mathcal{O}_U}(\widetilde{I_{A/B}}, \widetilde{I_{A/B}}) \\ &\cong H^1_*(U, \mathcal{H}om(\widetilde{I_{A/B}}, \widetilde{I_{A/B}})) \\ &\cong H^1_*(U, \widetilde{B}) \cong H^2_{I(Z)}(B) = 0\end{aligned}$$

and

$$_0\operatorname{Hom}(I_{A/B}, I_{A/B}) \cong H^0_*(U, \widetilde{B})_0 \cong B_0 \cong K,$$

which proves the claim. □

From the exact sequence (5.11), we get

$$\begin{aligned}&{}_0\operatorname{hom}_B(I_{A/B}, A) - {}_0\operatorname{ext}^1_B(I_{A/B}, A) \\ &\quad = {}_0\operatorname{hom}_B(I_{A/B}, B) - 1 - {}_0\operatorname{ext}^1_B(I_{A/B}, B) - r = b - 1 - r,\end{aligned}$$

where

$$r = \ker({}_0\operatorname{Ext}^2_B(I_{A/B}, I_{A/B}) \longrightarrow {}_0\operatorname{Ext}^2_B(I_{A/B}, I_B)) \leq e,$$

and we are done. □

To simplify the description of the formula in the lemma below, we define

$$\begin{aligned}s^2(\nu) &:= \dim(S^2K_B(2n+2+p))_\nu; \\ s^3(\nu) &:= \dim(S^3K_B(2n+2+p))_\nu\end{aligned}$$

and we denote the exact sequence (5.1) by

$$0 \longrightarrow \oplus_{i=1}^{m-1} R(-n_{2,i}) \longrightarrow \oplus_{i=1}^{m} R(-n_{1,i}) \longrightarrow I_B \longrightarrow 0. \tag{5.12}$$

We have the following lemma.

Lemma 5.3.3. *If* $\operatorname{codim}_R B = 2$ *and* $\operatorname{depth}_{I(Z)} B \geq 3$, *then*

$$a = \sum_{i=1}^{m} s^2(n_{1,i}) - \sum_{i=1}^{m-1} s^2(n_{2,i}) + s^3(n+1).$$

Proof. The exact sequence (5.12) leads to

$$0 \longrightarrow H_1 \longrightarrow \oplus_{i=1}^{m} B(-n_{1,i}) \longrightarrow I_B/I_B^2 \longrightarrow 0, \tag{5.13}$$

where H_1 is the Koszul homology. Since I_B/I_B^2 has rank 2, H_1 has rank 1 and the exact sequence (5.13) becomes

$$0 \longrightarrow K_B\left(n+1-\sum_{i=1}^{m} n_{1,i}\right) \longrightarrow \oplus_{i=1}^{t} B(-n_{1,i}) \longrightarrow I_B/I_B^2 \longrightarrow 0. \tag{5.14}$$

Indeed, let $U = \operatorname{Proj}(B) - Z$ be a set where B is a local complete intersection. It is well known that $\wedge^2(\widetilde{I_B/I_B^2})^*_{|U} \cong \widetilde{K_B}(n+1)$, and hence by (5.13) that

$$\widetilde{H_1}_{|U} \cong \wedge(\oplus_{i=1}^m \widetilde{B}(-n_{1,i})) \otimes \wedge^2(\widetilde{I_B/I_B^2})^*_{|U} \cong \widetilde{K_B}\left(n+1-\sum_{i=1}^m n_{1,i}\right)_{|U}.$$

Therefore, we have

$$\begin{aligned} H_1 \cong H^0_*(U, \widetilde{H_1}) &\cong H^0_*\left(U, \widetilde{K_B}\left(n+1-\sum_{i=1}^m n_{1,i}\right)\right) \\ &\cong K_B\left(n+1-\sum_{i=1}^m n_{1,i}\right). \end{aligned}$$

Moreover, there is an exact sequence (see [59, Lemma 4.9])

$$0 \longrightarrow K_B^* \longrightarrow \oplus_{i=1}^{m-1} B(-a_i + a_m - \ell) = \oplus_{i=1}^{m-1} B(-n_{2,i}) \longrightarrow H_1 \longrightarrow 0. \quad (5.15)$$

Applying $\operatorname{Hom}_B(-, S^2K_B(2n+2+p))$ to the exact sequence (5.13) we get

$$\begin{aligned} 0 &\longrightarrow {}_0\operatorname{Hom}_B(I_B/I_B^2, S^2K_B(2n+2+p)) \longrightarrow \oplus_{i=1}^m S^2K_B(2n+2+p+n_{1,i}) \\ &\longrightarrow {}_0\operatorname{Hom}_B(H_1, S^2K_B(2n+2+p)) \longrightarrow {}_0\operatorname{Ext}^1_B(I_B/I_B^2, S^2K_B(2n+2+p)) \\ &\longrightarrow 0 \end{aligned}$$

and

$${}_0\operatorname{Ext}^1_B(H_1, S^2K_B(2n+2+p)) \cong {}_0\operatorname{Ext}^2_B(I_B/I_B^2, S^2K_B(2n+2+p)).$$

Using the exact sequence (5.15), we obtain

$$\begin{aligned} 0 &\longrightarrow {}_0\operatorname{Hom}_B(H_1, S^2K_B(2n+2+p)) \longrightarrow \oplus_{i=1}^{m-1} S^2K_B(2n+2+p+n_{2,i}) \\ &\longrightarrow {}_0\operatorname{Hom}_B(K_B^*(-n-1), S^2K_B(2n+2+p)) \cong S^3K_B(3n+3+p) \\ &\longrightarrow {}_0\operatorname{Ext}^1_B(H_1, S^2K_B(2n+2+p)) \longrightarrow 0. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} a &= {}_0\hom_B(I_B/I_B^2, I_{A/B}) - {}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) + {}_0\operatorname{ext}^2_B(I_B/I_B^2, I_{A/B}) \\ &= \sum_{i=1}^m s^2(n_{1,i}) - \sum_{i=1}^{m-1} s^2(n_{2,i}) + s^3(n+1), \end{aligned}$$

which proves what we want. □

Lemma 5.3.4. *If* $\operatorname{codim}_R B = 2$ *and* $\operatorname{depth}_{I(Z)} B \geq 3$, *then*

$$b = {}_0\hom_B(S^3K_B(2n+2+p), K_B) - {}_0\operatorname{ext}^1_B(S^3K_B(2n+2+p), K_B).$$

Proof. Since $\operatorname{depth}_{I(Z)} B \geq 3$ and $\operatorname{depth}_{I(Z)} K_B \geq 3$, we have

$$\begin{aligned}\operatorname{Ext}^1_B(S^3K_B(2n+2+p),B) &\cong \operatorname{Ext}^1_{\mathcal{O}_U}(\widetilde{S^2K_B}(2n+2+p),\widetilde{B})\\ &\cong H^1_*(U,\mathcal{H}om(\widetilde{S^2K_B}(2n+2+p),\widetilde{B}))\\ &\cong H^1_*(U,\mathcal{H}om(\widetilde{S^3K_B}(2n+2+p),\widetilde{K_B}))\\ &\cong \operatorname{Ext}^1_B(S^3K_B(2n+2+p),B). \qquad \square\end{aligned}$$

We are now ready to write down bounds for $\dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ in terms of $a_1,\dots,a_m$ and $e := {}_0\operatorname{ext}^2_B(I_{A/B},I_{A/B})$ (resp., $a^0 := {}_0\hom_B(I_B/I_B^2,I_{A/B})$). Note that, by Lemma 5.3.1, a^0 (resp., a) vanishes if $a_m > 2a_{m-1}-a_1$ (resp., $a_m > 3a_{m-1}$).

Theorem 5.3.5. *Let $X \subset \mathbb{P}^n$ be a codimension 3, symmetric determinantal scheme defined by the submaximal minors of an $m\times m$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\dots,m}$, where $f_{ji} \in K[x_0,\dots,x_n]$ is a homogeneous polynomial of degree $a_j + a_i$. Let $\mathcal{B}$ be the matrix obtained by deleting the last row, let $Y = \operatorname{Proj}(B) \subset \mathbb{P}^n$ be the codimension 2, ACM scheme defined by the maximal minors of $\mathcal{B}$, and suppose $U = Y - Z \hookrightarrow \mathbb{P}^n$ is a local complete intersection for some closed subset Z such that $\operatorname{depth}_{I(Z)} B \geq 3$. Then,*

$$\begin{aligned}&\dim_{[X]}\operatorname{Hilb}^{p(t)}(\mathbb{P}^n)\\ &\geq \sum_{1\le i\le j\le k\le m-1}\binom{a_m-a_i-a_j-a_k+n}{n}\\ &\quad-\sum_{\substack{1\le i\le m\\1\le j\le k\le m-1}}\binom{a_m+a_i-a_k-a_j+n}{n}+\sum_{\substack{1\le i<j\le m\\1\le k\le m-1}}\binom{a_m+a_j+a_i-a_k+n}{n}\\ &\quad-\sum_{1\le i<j<k\le m}\binom{a_m+a_j+a_i+a_k+n}{n}+\sum_{\substack{1\le i\le m\\1\le j\le m-1}}\binom{a_i+a_j+n}{n}\\ &\quad-\sum_{1\le i\le j\le m-1}\binom{a_j-a_i+n}{n}-e-\sum_{1\le i\le j\le k\le m-1}\binom{a_i+a_j+a_k-a_m+n}{n}\\ &\quad+\sum_{\substack{1\le k,j\le m\\1\le i\le m-1}}\binom{a_i-a_j-a_k-a_m+n}{n}+\sum_{\substack{1\le k\le m-1\\1\le i\le j\le m-1}}\binom{a_i+a_j+a_k-a_m+n}{n}\\ &\quad-\sum_{1\le i,j\le m}\binom{a_i-a_j+n}{n}-\sum_{\substack{1\le k,i\le m-1\\1\le j\le m}}\binom{a_i-a_j+a_k-a_m+n}{n}.\end{aligned}$$

Moreover, if $\operatorname{Char}(K)\neq 2$, *then*

$$\dim_{[X]}\operatorname{Hilb}^{p(t)}(\mathbb{P}^n) \geq b-1+\dim(N_B)_0 - a^0.$$

Proof. By Proposition 5.3.2, $b-1-e+\dim(N_B)_0-a \le \dim_{[X]} \operatorname{Hilb}^{p(t)} \mathbb{P}^n$. So, we only need to compute b, a, and $\dim(N_B)_0$ in terms of $a_1,\dots,a_m$. To this end, we consider the exact sequence

$$\begin{aligned}0 \longrightarrow R \longrightarrow & \oplus_{i=1}^m R(-a_i-a_m+\ell) \\ & \xrightarrow{\mathcal{N}} \oplus_{i=1}^{m-1} R(a_i-a_m+\ell) \longrightarrow K_B(n+1) \\ & \longrightarrow 0.\end{aligned}$$

By [22, Theorem A2.10], S^2K_B and S^3K_B have a minimal free R-resolution of the following type:

$$\begin{aligned}0 & \longrightarrow \oplus_{1\le i<j\le m} R(-\ell-a_i-a_j) \longrightarrow \oplus_{\substack{1\le i\le m-1\\ 1\le j\le m}} R(-\ell+a_i-a_j) \\ & \xrightarrow{\mathcal{N}} \oplus_{1\le i<j\le m} R(-\ell+a_i+a_j) \longrightarrow S^2K_B(2n+2+p) \longrightarrow 0,\end{aligned} \tag{5.16}$$

$$\begin{aligned}0 & \longrightarrow \oplus_{1\le i<j<k\le m} R(3\ell-3a_m-a_i-a_j-a_k) \\ & \longrightarrow \oplus_{\substack{1\le k\le m-1\\ 1\le i<j\le m}} R(3\ell-3a_m-a_i-a_j+a_k) \\ & \longrightarrow \oplus_{\substack{1\le j\le k\le m-1\\ 1\le i\le m}} R(3\ell-3a_m-a_i+a_j+a_k) \\ & \longrightarrow \oplus_{1\le i\le j\le k\le m-1} R(3\ell-3a_m+a_i+a_j+a_k) \\ & \longrightarrow S^3K_B(3n+3) \longrightarrow 0.\end{aligned} \tag{5.17}$$

So, using Lemma 5.3.3 and the exact sequences (5.16) and (5.17), we get

$$\begin{aligned} a &= \sum_{i=1}^{m} s^2(n_{1,i}) - \sum_{i=1}^{m-1} s^2(n_{2,i}) + s^3(n+1) \\ &= \sum_{1\le i\le j\le k\le m-1} \binom{a_i+a_j+a_k-a_m+n}{n} + \sum_{\substack{1\le k,i\le m-1\\ 1\le j\le m}} \binom{a_i-a_j+a_k-a_t+n}{n} \\ &\quad - \sum_{\substack{1\le k,j\le m\\ 1\le i\le m-1}} \binom{a_i-a_j-a_k-a_m+n}{n} - \sum_{\substack{1\le k\le m-1\\ 1\le i\le j\le m-1}} \binom{a_i+a_j+a_k-a_m+n}{n}. \end{aligned} \tag{5.18}$$

One knows by [25] that

$$\begin{aligned}\dim(N_B)_0 = & \sum_{\substack{1\le i\le m\\ 1\le j\le m-1}} \binom{a_i+a_j+n}{n} - \sum_{1\le i,j\le m} \binom{a_i-a_j+n}{n} \\ & - \sum_{1\le i\le j\le m-1} \binom{a_j-a_i+n}{n} + 1.\end{aligned} \tag{5.19}$$

Finally, using Proposition 5.1.10 and Lemma 5.3.4 and applying the contravariant functor ${}_0\operatorname{Hom}(-, R(-n-1))$ to the exact sequence (5.17), we obtain

$$\begin{aligned} b &= {}_0\hom_B(I_{A/B}, B) - {}_0\operatorname{ext}^1_B(I_{A/B}, B) \qquad (5.20)\\ &= {}_0\operatorname{ext}^2_R(S^3K_B(2s+p), R(-n-1)) - {}_0\operatorname{ext}^3_R(S^3K_B(2s+p), R(-n-1))\\ &= \sum_{1\le i\le j\le k\le m-1}\binom{a_m - a_i - a_j - a_k + n}{n} - \sum_{\substack{1\le i\le m\\ 1\le j\le k\le m-1}}\binom{a_m + a_i - a_k - a_j + n}{n}\\ &\quad + \sum_{\substack{1\le i<j\le m\\ 1\le k\le m-1}}\binom{a_m + a_j + a_i - a_k + n}{n} - \sum_{1\le i<j<k\le m}\binom{a_m + a_j + a_i + a_k + n}{n}. \end{aligned}$$

Putting (5.18), (5.19) and (5.18) together, we get the first lower bound.

For the second lower bound, see [60, Theorem 4.6]. □

We are led to pose the following question.

Question 5.3.6. *Under which extra hypothesis the bounds given in Theorem* 5.3.5 *are sharp?*

We will now give some examples.

Example 5.3.7. Set $R = K[x_0, x_1, \dots, x_5]$. Let $X \subset \mathbb{P}^5 = \operatorname{Proj}(R)$ be the Veronese surface defined by the 2×2 minors of the homogeneous symmetric matrix

$$\mathcal{A} = \begin{pmatrix} x_0 & x_1 & x_2\\ x_1 & x_3 & x_4\\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let I_A be the ideal generated by the 2×2 minors of $\mathcal{A}$ and let I_B be the ideal generated by the 2×2 minors of

$$\mathcal{N} = \begin{pmatrix} x_0 & x_1 & x_2\\ x_1 & x_3 & x_4 \end{pmatrix}.$$

Set $A = R/I_A$ and $B = R/I_B$. It is well known that $\dim_{[X]}\operatorname{Hilb}^{p(t)}(\mathbb{P}^5) = 27$. If we apply Theorem 5.3.5, we obtain $\dim_{[X]}\operatorname{Hilb}^{p(t)}(\mathbb{P}^5) \ge 29 - e$ where $e = {}_0\operatorname{ext}^2_B(I_{A/B}, I_{A/B})$. Using Macaulay2 program [34], we have computed the dimension of ${}_0\operatorname{Ext}^2_B(I_{A/B}, I_{A/B})$ and we have got $e = 2$. Thus, $\dim_{[X]}\operatorname{Hilb}^{p(t)}(\mathbb{P}^5) \ge 27$ and hence the first bound given in Theorem 5.3.5 is sharp.

Example 5.3.8. Set $R = K[x_0, x_1, \dots, x_5]$. Let $X \subset \mathbb{P}^5 = \operatorname{Proj}(R)$ be the surface defined by the 3×3 minors of the homogeneous symmetric matrix

$$\mathcal{A} = \begin{pmatrix} x_0 & x_1 & x_2 & L_1\\ x_1 & x_3 & x_4 & L_2\\ x_2 & x_4 & x_5 & L_3\\ L_1 & L_2 & L_3 & L_4 \end{pmatrix},$$

where L_i are general linear forms. Let I_A be the ideal generated by the 3×3 minors of $\mathcal{A}$ and let I_B be the ideal generated by the 3×3 minors of

$$\mathcal{B} = \begin{pmatrix} x_0 & x_1 & x_2 & L_1 \\ x_1 & x_3 & x_4 & L_2 \\ x_2 & x_4 & x_5 & L_3 \end{pmatrix}.$$

Set $A = R/I_A$ and set $B = R/I_B$.

If we apply Theorem 5.3.5, we obtain $\dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^5) \geq 59 - e$ where $e = {}_0\operatorname{ext}^2_B(I_{A/B}, I_{A/B})$. Using the Macaulay program [6], we have computed the dimension of ${}_0\operatorname{Ext}^2_B(I_{A/B}, I_{A/B})$ and we have got $e = 14$. Thus, $\dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^5) \geq 45$. Using again the Macaulay program, we have computed $h^0(\mathcal{N}_X)$ and we have got $h^0(\mathcal{N}_X) = 45$. Hence, the first bound given in Theorem 5.3.5 is sharp.

The results proved in this last section give rise to a number of quite interesting questions and possible generalizations that we gather in this part of the work. In fact, Examples 5.3.7 and 5.3.8 suggest – and prove for $t \leq 4$ – the following question.

Question 5.3.9. *Let $\mathcal{A} = (f_{ji})_{i,j=1,\dots,m}$ be an $m \times m$ homogeneous symmetric matrix, where $f_{ji} \in K[x_0, x_1, \dots, x_n]$ are general linear forms. Let $X \subset \mathbb{P}^n$ be a codimension 3, symmetric determinantal subscheme defined by the submaximal minors of $\mathcal{A}$. Is it true that*

$$\begin{aligned} \dim_{[X]} \operatorname{Hilb}^{p(t)}(\mathbb{P}^n) = {} & (n+1)m[(m-1)^2 + (m-1)] \\ & - m^2 - (m-1)(m^2-1) - m\binom{m}{2} \\ & - \binom{n+2}{2}\binom{m}{3} - (n+1)\binom{m+1}{3} - e \quad ? \end{aligned}$$

More generally, we would like to know if under certain numerical conditions on $a_1, \dots, a_m$ the bounds given in Theorem 5.3.5 are sharp. So, we are led to pose the following problem.

Problem 5.3.10. *Find numerical conditions on the rational numbers $a_1, \dots, a_m$ which allow us to assure that the bounds given in Theorem* 5.3.5 *are sharp.*

As above, we denote by $S(a_1, \dots, a_m)$ the irreducible family of codimension 3, symmetric determinantal subschemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix $\mathcal{A} = (f_{ji})_{i,j=1,\dots,m}$, where $f_{ji} \in K[x_0, x_1, \dots, x_n]$ are homogeneous polynomials of degree $a_i + a_j \in \mathbb{Z}_+$. Let $\mathcal{B}$ be the matrix obtained by deleting the last row of $\mathcal{A}$ and let $Y \subset \mathbb{P}^n$ be the codimension 2, ACM scheme defined by the maximal minors of $\mathcal{B}$. Denote by $S(\mathcal{B})$ the part of $\operatorname{Hilb}^{p(t)}(\mathbb{P}^n)$ consisting of those Y' obtained by deforming $\mathcal{B}$ as a matrix with symmetric "left" $(m-1) \times (m-1)$ matrix.

Problem 5.3.11. *Find an explicit formula for*

$$\dim_{[X]} S(a_1, \ldots, a_m) \qquad \textit{and} \qquad \dim_{[Y]} S(\mathcal{B}).$$

Problem 5.3.12. *Is* $\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n) = \dim_{[X]} S(a_1, \ldots, a_m)$*?*

The computations we have made, using Macaulay 2 [34], suggest the following conjecture.

Conjecture 5.3.13. *If* $m = 3$ *and* $2a_i = p$ *for all* i*, then*

$$\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n) = \dim_{[X]} S(a_1, \ldots, a_m) = h^0(\mathcal{N}_X) = 6\binom{p+n}{n} - 9.$$

We have checked that the conjecture is true for $n = 5$ and $1 \leq p \leq 10$.

Bibliography

[1] S.S. Abhyankar and D.V. Kulkarni, *Coinsertion and standard bitableaux*, Discrete Math. **85** (1990), 115–166.

[2] S.S. Abhyankar and D.V. Kulkarni, *Bijection between indexed monomials and standard bitableaux*, Discrete Math. **79** (1990), 1–48.

[3] R. Apéry, *Sur certain caractères numériques d'un idéal sans composant impropre*, C. R. Acad. Sci. Paris **220** (1945), 234–236.

[4] L. Avramov and J. Herzog, *The Koszul algebra of a codimension* 2 *embedding*, Math. Z. **175** (1980), 249–260.

[5] C. Baetica, *Rees algebra of ideals generated by Pfaffians*, Comm. Algebra **26** (1998), 1769–1778.

[6] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra, available at ftp://math.harvard.edu via anonymous ftp.

[7] V. Beorchia and Ph. Ellia, *Normal bundle and complete intersections*, Rend. Sem. Mat. Univ. Politec. Torino **48** (1990), 553–562.

[8] W. Bruns, *The Eisenbud–Evans generalized principal ideal theorem and determinantal ideals*, Proc. Amer. Math. Soc. **83** (1981), 19–24.

[9] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Stud. Adv. Math. **39**, Cambridge University Press, Cambridge, 1993.

[10] W. Bruns and U. Vetter, *Determinantal rings*, Lectures Notes in Math. **1327**, Springer-Verlag, New York/Berlin, 1988.

[11] N. Budur, M. Casanellas, and E. Gorla, *Hilbert functions of irreducible arithmetically Gorenstein schemes*, J. Algebra **272** (2004), 292–310.

[12] M.C. Chang, *A filtered Bertini-type theorem*, J. Reine Angew. Math. **397** (1989), 214–219.

[13] M. Casanellas and R.M. Miró–Roig, *Gorenstein liaison of divisors on standard determinantal schemes and on rational normal scrolls*, J. Pure Appl. Algebra **164** (2001), 325–343.

[14] M. Casanellas and R.M. Miró–Roig, *Gorenstein liaison of curves in* $\mathbb{P}^4$, J. Algebra **230** (2000), 656–664.

[15] M. Casanellas and R.M. Miró–Roig, *Gorenstein liaison and special linear configurations*, Illinois J. Math. **46** (2002), 129–143.

[16] G. Casnati and F. Catanese, *Quadratic sheaves and self-linkage*, Asian J. Math. **6** (2002), 731–742.

[17] A. Conca, *Gröbner bases of ideals of minors of a symmetric matrix*, J. Algebra **166** (1994), 406–421.

[18] E. De Negri, *ASL and Gröbner Bases Theory for Pfaffians and Monomial Algebras*, Dissertation, Universität Essen (1996).

[19] P. Dubreil, *Quelques propriétés des variétés algébriques*, Actualités Sci. Industr. **210**, Hermann, Paris, 1935.

[20] J.A. Eagon and M. Hochster, *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

[21] J.A. Eagon and D.G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. London **269** (1962), 188–204.

[22] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Grad. Texts in Math. **150** , Springer-Verlag, 1995.

[23] D. Eisenbud, *The geometry of syzygies*, Grad. Texts in Math. **229**, Springer-Verlag, 2005.

[24] D. Eisenbud and B. Mazur, *Evolutions, symbolic powers, and Fitting ideals*, J. Reine Angew. Math. **488** (1997), 189–210.

[25] G. Ellingsrud, *Sur le schéma de Hilbert des variétés de codimension* 2 *dans* $\mathbb{P}^e$ *a cône de Cohen–Macaulay*, Ann. Scient. École Norm. Sup. (4) **8** (1975), 423–432.

[26] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

[27] D. Franco, S. Kleiman, and A. Lascu, *Gherardelli linkage and complete intersections*, Michigan Math. J. **48** (2000), 271–279.

[28] F. Gaeta, *Sulle curve sghembe algebriche di residuale finito*, Ann. Mate. Pura Appl. (4) **27** (1948), 177–241.

[29] L. Gold, H. Schenk, and H. Srinivasan, *Betti numbers and degree bounds for some linked zero-schemes*, J. Pure Appl. Alg. **210** (2007), 481–491.

[30] E.S. Golod, *A remark on perfect ideals* (*Russian*), Algebra Collect. (1990), 37–39, Zentralblatt 496, 13009.

[31] E. Gorla, *The G-biliaison class of symmetric determinantal schemes*, J. Algebra **310** (2007), 880–902.

[32] E. Gorla, *A generalized Gaeta's theorem*, preprint math.AG/0701456.

[33] E. Gorla, J. Migliore, R.M. Miró–Roig, and U. Nagel, *Notes on the multiplicity conjecture*, work in progress.

[34] D. Grayson and M. Stillman, Macaulay 2 – A software system for algebraic geometry and commutative algebra,
available at http://www.math.uiuc.edu/Macaulay2/.

[35] A. Grothendieck, *Les schémas de Hilbert*, Séminaire Bourbaki, Exp. **221** (1960).

[36] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et Théorèmes de Lefschetz locaux et globaux*, North Holland, Amsterdam, 1968.

[37] E. Guardo and A. Van Tuyl, *Powers of complete intersections: Graded Betti numbers and applications*, Illinois J. Math., **49** (2005), 265–279.

[38] R. Hartshorne, *Connectedness of the Hilbert scheme*, Publ. Math. Inst. Hautes Études Sci. **29** (1966), 261–304.

[39] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. **52**, Springer-Verlag, New York, 1983.

[40] R. Hartshorne, *Lectures on deformation theory*, preprint, 2004.

[41] R. Hartshorne, *Some examples of Gorenstein liaison in codimension* 3, Collect. Math. **53** (2002), 21–48.

[42] R. Hartshorne, *Generalized divisors on Gorenstein schemes*, K-theory **8** (1994), 287–339.

[43] J. Herzog, *Deformationen von Cohen–Macaulay Algebren*, J. Reine Angew. Math. **318** (1980), 83–105.

[44] J. Herzog, *Ein Cohen–Macaulay-Kriterium mit Anwendungen auf den Konormalenmodul und den Differentialmodul*, Math. Z. **163** (1978), 345–360.

[45] J. Herzog and M. Kühl, *On the Betti numbers of finite pure and linear resolutions*, Comm. Algebra **12** (1984), 1627–1646.

[46] J. Herzog and H. Srinivasan, *Bounds for multiplicities*, Trans. Amer. Math. Soc. **350** (1998), 2879–2902.

[47] J. Herzog and N.V. Trung, *Gröbner bases and multiplicity of determinantal and Pfaffian ideals*, Adv. Math. **96** (1992), 1–37.

[48] J. Herzog, N.V. Trung, and G, Valla, *On hyperplane sections of reduced irreducible varieties of low codimension*, J. Math. Kyoto Univ. **34** (1994), 47–72.

[49] J. Herzog and X. Zheng, *Notes on the multiplicity conjecture*, Collect. Math. **57** (2006), 211–226.

[50] C. Huneke, *Linkage and the Koszul homology of ideals*, Amer. J. Math. **104** (1982), 1043–1062.

[51] C. Huneke and M. Miller *A note on the multiplicity of Cohen–Macaulay algebras with pure resolutions*, Canad. J. Math. **37** (1985), 1149–1162.

[52] C. Huneke and B. Ulrich, *The structure of linkage*, Ann. of Math. (2) **126** (1987), 277–334.

[53] T. Józefiak, *Ideals generated by minorts of a symmetric matrix*, Comment. Math. Helv. **53** (1978), 595–607.

[54] D. Laksov, *Deformation and transversality*, in Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen), Lecture Notes in Math. **732**, Springer-Verlag 1978, 300–316.

[55] S. Kleiman and B. Ulrich, *Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers*, Trans. Amer. Math. Soc. **349** (1997), 4973–5000.

[56] J.O. Kleppe, J.C. Migliore, R.M. Miró–Roig, U. Nagel, and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. **154** (2001), viii + 116.

[57] J.O. Kleppe and C. Peterson, *Maximal Cohen–Macaulay modules and Gorenstein algebras*, J. Algebra **238** (2001), 776–800.

[58] J.O. Kleppe and C. Peterson, *Sheaves with canonical determinant on Cohen–Macaulay schemes*, J. Algebra **256** (2002), 250–279.

[59] J.O. Kleppe and R.M. Miró–Roig, *Dimension of families of determinantal schemes*, Trans. Amer. Math. Soc. **357** (2005), 2871–2907.

[60] J.O. Kleppe and R.M. Miró–Roig, *Unobstructedness and dimension of families of codimension 3 ACM algebras*, Contemp. Math., to appear.

[61] J. Kreuzer, J. Migliore, U. Nagel, and C. Peterson, *Determinantal schemes and Buchsbaum–Rim sheaves*, J. Pure Appl. Algebra **150** (2000), 155–174.

[62] R.E. Kutz, *Cohen–Macaulay rings and ideal theory in rings of invariants of algebraic groups*, Trans. Amer. Math. Soc. **194** (1974), 115–129.

[63] F.S. Macaulay, *On a method of dealing with the intersection of plane curves*, Trans. Amer. Math. Soc. **5** (1904), 385–410.

[64] H. Matsumura, *Commutative algebra*, second ed., Benjamin/Cummings, Reading, 1980.

[65] J. Migliore, *On linking double lines*, Trans. Amer. Math. Soc. **294** (1986), 177–185.

[66] J. Migliore and U. Nagel, *Monomial ideals and the Gorenstein liaison class of a complete intersection*, Compositio Math. **133** (2002), 25–36.

[67] J. Migliore, U. Nagel, and T. Römer, *The multiplicity conjecture in low codimension*, Math. Res. Lett. **12** (2005), 731–747.

[68] R.M. Miró–Roig, *A note on the multiplicity of determinantal ideals* , J. Algebra **299** (2006), 714–724.

[69] R.M. Miró–Roig, *On the multiplicity conjecture*, Contemp. Math., to appear.

[70] R.M. Miró–Roig, *Lectures on determinantal ideals*, Notes of the School in Commutative Algebra, Mumbai, 2008.

[71] R.M. Miró–Roig, *Betti numbers of determinantal ideals*, J. Algebra, to appear.

[72] U. Nagel, *Even liaison classes generated by Gorenstein linkage*, J. Algebra **209** (1998), 543–584.

[73] M. Noether, *Zur Grundlegung der Theorie der algebraischen Raumkurven*, Verl. König. Akad. Wiss., Berlin, 1883.

[74] C. Okonek, *Reflexive Garben auf* $\mathbb{P}^4$, Math. Ann., **260** (1982), 211–237.

[75] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Invent. Math. **26** (1974), 271–302.

[76] A.P. Rao, *On self-linked curves*, Duke Math. J. **49** (1982), 251–273.

[77] A.P. Rao, *Liaison among curves in* $\mathbb{P}^3$, Invent. Math. **50** (1979), 205–217.

[78] T. Römer, *Betti numbers and shifts in minimal graded free resolutions*, preprint, arXiv:math.AC/070119.

[79] P. Schenzel, *Notes on liaison and duality*, J. Math. Kyoto Univ. **22** (1982), 485–498.

[80] P. Schwartau, *Liaison Addition and Monomial Ideals*, Ph.D. Thesis, Brandeis University, 1982.

[81] E. Sernesi, *Topics on families of projective schemes*, Queen's Papers in Pure and Appl. Math. **73**, Queen's University Kongston, 1986.

[82] F. Severi, *Sulle classificazione delle curve algebraiche e sul teoremo d'esistenza di Riemann.* Rend. R. Acc. Naz. dei Lincei **5** 24, I (1915), 877–888, 1011–1020.

[83] F. Severi, *Memorie scelte*, C. Zuffi, Bologna (1950).

[84] J. Stückrad and W. Vogel, *Buchsbaum rings and applications. An interaction between algebra, geometry and topology*, Springer-Verlag, Berlin, 1986.

[85] B. Sturmfels, *Gröbner bases and Stanley decompositions of determinantal rings*, Math. Z. **205** (1990), 137–144.

[86] B. Ulrich, *Ring of invariants and linkage of determinantal ideals* , Math. Ann. **274** (1986), 1–17.

This page intentionally left blank

Index

$A(V)$, 1
$\beta_{i,j}(M)$, 5
$e(M)$, 9
$e(R/I)$, 5
$\Gamma_J(-)$, 2
$H^0_{\mathfrak{m}}(M)$, 2
$H^i_J(-)$, 2
$H^i_{\mathfrak{m}}(-)$, 2
$H^i_*(X, \mathcal{F})$, 2
h-vector, 10
$\mathrm{Hilb}^{p(t)}(\mathbb{P}^r)$, 63
$\underline{\mathrm{Hilb}}^r_{p(t)}$, 63
$I_j(\mathcal{A})$, 14
$I^s_{m,t}$, 14
$I_{p,q,r}$, 14
K_X, 11, 12
$M^i(X)$, 7
$\mu(I)$, 17
$\tilde{M}$, 2
$\mathcal{N}_V$, 1
ω_X, 11
$\mathrm{pd}(M)$, 4
$\mathrm{reg}(M)$, 5
$S(a_1, \ldots, a_m)$, 121
$\mathrm{Tor}^R_i(M, K)$, 5
$W(\underline{b}; \underline{a})$, 20, 21, 67, 79, 80, 83, 86, 88, 92, 93, 97, 98, 100, 101
$W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$, xiv, 20, 64, 76
$W_s(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$, 20, 64, 76

ACM, 6, 35, 65
AG, 11
almost linear resolution, 39
arithmetically Cohen–Macaulay, 6
  codimension 2 ∼, 33
  subschemes, 36
arithmetically Gorenstein, 11
  codimension 2 ∼, 12
Artinian reduction, 10, 47
Auslander–Buchsbaum formula, 6

basic double G-link, 52, 53
Betti numbers, xiv, 5, 47
  graded ∼, 5, 45, 47, 112
  total ∼, 5
Bordiga surface, 28, 41
Buchsbaum–Rim
  complex, xiv, 18, 19
  module, 73
  resolution, 77, 80, 81
  sheaf, 19

canonical module, 11, 12, 36
canonical sheaf, 11
Castelnuovo surface, 28, 39, 41
Castelnuovo–Mumford regularity, 5, 10
Cayley–Bacharach property, 13
CI-bilinked, 23
CI-liaison, xi, 21, 23, 105
CI-liaison invariant, 34, 36
CI-linked, 22
  self-∼, 23
Cohen–Macaulay module, 6
  maximal ∼, 86
Cohen–Macaulay type, 7, 8, 11
cohomology modules, 2
collinear points, 6

complete intersections, 7, 12, 15, 17, 34, 35, 76
  of type, 8
componentwise linear, 58
condition G_r, 26
cyclic, 12

deficiency module, 7, 31, 34
degeneracy locus, 19
degree matrix, 19, 47, 48, 51, 108, 112
depth, 2
$\operatorname{depth}_J M$, 2
$\operatorname{depth} M$, 2
determinantal ideal, xi, 14, 106, 111
  standard ∼, 14, 54
  symmetric ∼, 106, 111, 115
determinantal scheme, 15, 107
  flag of a ∼, 20
  good ∼, 15, 20, 67, 76, 98
  standard ∼, 15, 20, 67, 76
  symmetric ∼, 16, 107
determinantal standard, 46
$\dim M$, 2

Eagon–Northcott
  complex, xiv, 18, 19, 46, 51, 67, 80
  resolution, 83, 89, 92
Ellingsrud's theorem, 65
elliptic quintic, 13, 24
embedding codimension, 10

flag of determinantal schemes, 20, 94

G-bilinked, 23, 27
G-liaison, xi, 21, 23, 105
G-liaison invariant, 34, 40
G-link
  basic double ∼, 52, 53
G-linked, 22
  self-∼, 23
Gaeta's theorem, xii, 29, 33, 34, 40, 43, 106

generalized Koszul complexes, 17
geometrically
  sum of ∼ linked ideals, 25
geometrically CI-linked, 22
geometrically G-linked, 22
glicci, xiii, xv, 28, 40, 41, 43, 45, 106, 107
glicci scheme, 24
Gorenstein in codimension $\leq r$, 26
graded Betti numbers, 45, 47, 112
Grassmannian, 76
Grothendieck's theorem, 63

Hartshorne–Rao module, 7, 31
Hartshorne–Schenzel theorem, 31
Herzog–Huneke–Srinivasan conjecture, 46
Hilbert flag scheme, xiv, 79, 94
Hilbert function, 9, 31
Hilbert functor, 63
Hilbert polynomial, 5, 9, 13, 28, 31, 40, 45, 94
Hilbert scheme, xiv, 28, 63–67, 94, 96, 106
Hilbert syzygy theorem, 4
Hilbert–Burch
  matrix, 18, 33, 47, 65
  theorem, 18, 29, 47
homogeneous matrix, 14
  t-∼, 14
Horseshoe lemma, 27
hypersurface section, 26

Józefiak's resolution, 109
Józefiak's theorem, 112

k-syzygy, 2
  minimal ∼, 3
Knuth–Robinson–Schensted correspondence, xi
Koszul complexes
  generalized ∼, 17
Koszul homology, 86
Koszul resolution, 8, 12
Krull dimension, 2

licci, xii, 29, 33, 36, 38, 39, 41, 43, 45
licci scheme, 24, 36, 38
linear resolution, 29, 39, 40, 46, 106
local cohomology, 2, 36

mapping cone, 25, 31
maximal Cohen–Macaulay module, 86
minimal effective divisor, 27
minimal free resolution, 1, 3, 31, 113, 114, 116
minimal k-syzygy, 3
minimal free R-resolution, 46
moduli functor, 63
multiplicity, 5, 9
multiplicity conjecture, xii, xiii, xv, 46, 47, 54, 105, 111

Nakayama's lemma, 3
normal module, 70, 98
normal sheaf, 1, 64

pfaffians, 24
projective dimension, 4
pure resolution, xiii, 45, 46, 48, 54, 58, 115

quasi-pure resolution, xiii, 46, 58

Rao module, 7
rational normal curve, 9, 11, 15, 43, 45
rational normal quartic, 8, 23, 24, 39, 41
rational normal scroll, 76, 108
rational quartic, 32
resolution
  almost linear $\sim$, 39
  Buchsbaum–Rim $\sim$, 77, 80, 81
  Eagon–Northcott $\sim$, 83, 89, 92
  Józefiak's $\sim$, 109
  linear $\sim$, 29, 39, 40, 46, 106
  minimal free $\sim$, 3, 31, 113, 114, 116
  minimal free R-$\sim$, 46
  pure $\sim$, xiii, 45, 46, 48, 54, 58, 115
  quasi-pure $\sim$, xiii, 46, 58
Riemann–Roch's theorem, 39

scheme, 1
  arithmetically Cohen–Macaulay $\sim$, 6
  arithmetically Gorenstein $\sim$, 11
  complete intersection $\sim$, 7, 34, 35
  determinental $\sim$, 15
  glicci $\sim$, 24
  Hilbert $\sim$, xiv, 28, 63–67, 94, 96, 106
  Hilbert flag $\sim$, xiv, 79, 94
  licci $\sim$, 24
Schwartau, 34
self-CI-linked, 23
self-G-linked, 23
sheaf cohomology, 2
skew symmetric matrix, 24
smooth rational normal curve, 76
snake lemma, 90
spectral sequence, 36
standard determinantal, 41, 43, 54, 59
standard determinantal ideal, xi, 106
standard determinantal locus, 79
subladder, 107
sum of geometrically linked ideals, 25
symmetric determinantal ideal, xi, 14, 106, 111, 115
symmetric determinantal scheme, 107
symmetric matrix, 110, 111

theorem
  Ellingsrud's $\sim$, 65
  Gaeta's $\sim$, xii, 29, 33, 34, 40, 43, 106
  Grothendieck's $\sim$, 63

Hartshorne–Schenzel ∼, 31
Hilbert syzygy ∼, 4
Hilbert–Burch ∼, 18, 29, 47
Józefiak's ∼, 112
Riemann–Roch's ∼, 39
twisted cubic, 8, 23, 24, 66

unobstructed, xiv, 65, 94, 98
unobstructedness, xii, xv, 65, 97, 105

Veronese surface, 16, 107, 111, 119
Veronese variety, 10